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ON KANNAN-GERAGHTY MAPS AS AN EXTENSION OF KANNAN MAPS

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ABSTRACT. Extending the concept of weakly Kannan maps on metric spaces, we study the maps as $f : X \to X$ on a metric space (X, d) satisfying condition $d(f(x), f(y)) \leq (1/2)\beta(d(x, y))[d(x, f(x)) + d(y, f(y))]$ for every $x, y \in X$ and a function $\beta : [0, \infty) \to [0, 1)$ where for every sequence $t = \{t_n\}$ of non-negative real numbers satisfying $\beta(t_n) \to 1$, while $t_n \to 0$. Such a map is named the Kannan-Geraghty map because of its relation to weakly Kannan map and Geraghty contraction. Firstly, we show that our new condition is different from weakly Kannan condition. Having proven the fixed point theorem, we present two useful results on Kannan-Geraghty maps. Also, we illustrate some examples of Kannan-Graghty map having interesting properties.

1. INTRODUCTION

In 1968, R. Kannan started the study of fixed point theory on some contractive maps. A map $f : X \to X$ on a metric space (X, d) is said to be *contractive* if it satisfies the condition $d(f(x), f(y)) \leq qd(x, y)$ for any $x, y \in X$ and a fixed real number $q \in [0, 1)$. If the coefficient q (instead of a constant number) be a function as $q : X \times X \to [0, 1)$ satisfying the condition $sup\{q(x, y)|x, y \in X, a \leq d(x, y) \leq b\} < 1$ for every positive real numbers a and b (with $a \leq b$), then f is said to be *weakly contractive.*

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The well known Kannan Theorem was a variant version of the Banach contraction principle ([5]). The Banach contraction principle says that every contractive map on a complete metric space has a unique fixed point. In [1] Ruis and Melando extended Kannan theorem to the class of weakly Kannan maps, and then they gave a continuation method for this class. Recently, the single and set-valued α-η-ψ-contractive mappings have been studied in [4].

In this paper, based on the articles [1] and [8], we present the concept of Kannan-Geraghty maps, and we prove that Kannan-Graghty self mapping has a unique fixed point, and also Kannan-Geraghty non-self mapping has a best proximity point. Then we show two theorems; in the first theorem, we show the relation between weakly Kannan and Kannan Geraghty and in the second, relation between Kannan-Geraghty and weakly Kannan mappings.

2. Preliminaries

In this section, we recall some basic notations, definitions and theorems from references [7, 2, 5, 1, 8]. We discuss on the class Γ consisting of all of functions $\beta : [0, \infty) \to [0, 1)$ such that for every convergent sequence $t = \{t_n\}$ of non-negative real numbers satisfying $\beta(t_n) \to 1$ while $t_n \to 0$.

Definition 2.1. *([2]) Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be a Geraghty-contraction if it satisfies the condition $d(f(x), f(y)) \le \beta(d(x, y))d(x, y)$ for a continuous function $\beta \in \Gamma$.*

Theorem 2.1. ([2]) Let $d(X, d)$ be a complete metric space and $f : X \to X$ be a Geraghty-contraction. Then, f has a unique fixed point.

Following the Geraghty notations, for any two disjoint sequences $\bar{x} = \{x_n\}$ and $\bar{y} = \{y_n\}$ of points in a metric space (X, d) (s.t. $x_n \neq y_n$ for $n = 1, 2, 3, \cdots$), we use two sequences of non-negative real numbers $\bar{\delta}(\bar{x}, \bar{y}) := \{\delta_n(\bar{x}, \bar{y})\}$ and $\bar{\Delta}(\bar{x}, \bar{y}) := \{\Delta_n(\bar{x}, \bar{y})\}$ defined by

$$\delta_n(\bar{x}, \bar{y}) = d(x_n, y_n) \text{ and } \Delta_n(\bar{x}, \bar{y}) = \frac{d(f(x_n), f(y_n))}{d(x_n, y_n)}.$$

Theorem 2.2. ([5]) Let $f : X \to X$ be a contractive mapping on a complete metric space (X, d) and take $x_0 \in X$ and $x_n = f(x_{n-1})$ for $n = 1, 2, 3, \cdots$. Then, $x_n \to x_\infty$ in X, where x_∞ is the unique fixed point of f, if and only if for any two subsequences $\hat{x} := \{x_{h_n}\}$ and $\tilde{x} := \{x_{k_n}\}$ (where $x_{h_n} \neq x_{k_n}$ for $n = 1, 2, 3, \cdots$) we have

$$\Delta_n(\hat{x}, \tilde{x}) \to 1 \Rightarrow d_n(\hat{x}, \tilde{x}) \to 0.$$

Definition 2.2. *([9]) Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be weakly Kannan if it satisfies the condition*

$$d(f(x), f(y)) \leq \frac{\overline{\alpha}(x, y)}{2}[d(x, f(x)) + d(y, f(y))]$$

for every points $x, y \in X$, where $\overline{\alpha} : X \times X \to [0, 1)$ is a real-valued function satisfying the condition

$$\Theta(a, b) := \sup\{\overline{\alpha}(x, y) : x, y \in X \text{ and } a \leq d(x, y) \leq b\} < 1$$

for every positive real numbers $a \leq b$.

Theorem 2.3. ([1]) Let (X, d) be a complete metric space. If $f : X \to X$ is a weakly Kannan mapping, then f has a unique fixed point x^* and the Picard sequence of iterates $\{f^n(x)\}_{n \in N}$ converges to x^* for every $x \in X$.

Now, let A, B be two nonempty subsets of a metric space (X, d). The subsets A_0 and B_0 of A and B (respectively) are defined as follow:

$$A_0 := \{x \in A : d(x, y) = d(A, B), \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = d(A, B), \text{ for some } x \in A\},$$

where $d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$.

Definition 2.3. *([9]) Let $A, B \subset X$ be two nonempty subsets of a metric space (X, d) and $f : A \to B$ be an arbitrary mapping. An element $x \in A$ is said to be a best proximity point of the mapping f if it satisfies the equality $d(x, f(x)) = d(A, B)$.*

Definition 2.4. *([9]) Assume that $A, B \subset X$ be two nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$.*

(i) The pair (A, B) is said to have P-property if for any points $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, we have:

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

(ii) The pair (A, B) is said to have weak P-property if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$, we have:

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) \leq d(y_1, y_2)$$

Here, we remember the straightforward generalization of the concept of weakly Kannan map and Geraghty contraction to the non-self-mapping case.

Definition 2.5. *([9]) Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d). A map $f : A \to B$ is said to be weakly Kannan if it satisfies the inequality*

$$d(f(x), f(y)) \leq \frac{\overline{\alpha}(x, y)}{2}[d(x, f(x)) + d(y + f(y)) - 2d(A, B)]$$

for every $x, y \in X$ and a real-valued function $\overline{\alpha} : X \times X \to [0, 1)$ such that $\Theta(a, b) :=$ $\sup\{\overline{\alpha}(x, y) : a \leq d(x, y) \leq b\} \leq 1$ for every real numbers $0 < a \leq b$.

Definition 2.6. *([2]) Let A, B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a Geraghty contraction if there exists $\beta \in \Gamma$ such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$$

for every $x, y \in A$.

Notice that since $\beta : [0, \infty) \to [0, 1)$, we have

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) < d(x, y),$$

Therefore, every Geraghty-contraction is a contractive mapping.

Finally, we introduce two new versions of contractive mappings, namely *Kannan-Geraghty maps* separately in selfmapping and non-selfmapping cases, on which we will prove fixed point theorem in the next section.

Definition 2.7. *Let (X, d) be a metric space. A mapping $f : X \to X$, is said to be a selfmapping Kannan Geraghty map if there exists a real valued function $\beta \in \Gamma$ such that, for all $x, y \in X$ we have*

$$d(f(x), f(y)) \leq \frac{\beta(d(x, y))}{2}[d(x, f(x)) + d(y, f(y))].$$

Definition 2.8. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space. A mapping $f : A \to B$ is said to be a non-selfmapping Kannan Geraghty map if there exists real valued function $\beta \in \Gamma$ where we have*

$$d(f(x), f(y)) \leq \frac{\beta(d(x, y))}{2}[d(x, f(x)) + d(y, f(y))] \tag{2.1}$$

for every $x, y \in A$.

3. Main Results

In this section, we prove some theorems among them Theorem 3.1, and theorem 3.2 is our main result. Also we give some examples.

Theorem 3.1. *Let $f : X \to X$ be a Kannan-Geraghty map on a complete metric space (X, d). Then, f has a unique fixed point $u \in X$ and for any $x_0 \in X$, the sequence of iterates $\{f^n(x_0)\}$ converges to u.*

Proof. Since $f : X \to X$ is Kannan-Geraghty mapping, there exists a function $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

Consider any $x_0 \in X$ and define $x_n = f(x_{n-1}), n = 1, 2, \ldots.$ We assume that $d(x_0, x_1) > 0$, otherwise there is nothing to prove. We prove that $d(x_n, x_{n+1}) \to 0$, and then, $\{x_n\}$ converges to a point u which is the unique fixed point of f.

From the following inequality

$$d(f(x_n), f(x_{n-1})) \leq \frac{\beta(d(x_n, x_{n-1}))}{2}[d(x_n, f(x_n)) + d(x_{n-1}, f(x_{n-1}))]$$

we have

$$d(x_{n+1}, x_n) \leq \frac{\beta(d(x_n, x_{n-1}))}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)],$$

which gives

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{\beta(d(x_n, x_{n-1}))}{2}[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)], \\ &\leq [\frac{1}{2}d(x_n, x_{n+1}) + \frac{\beta(d(x_n, x_{n-1}))}{2}d(x_{n-1}, x_n)], \end{aligned}$$

and hence, we have

$$d(x_{n+1}, x_n) \leq \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1}),$$

then

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}). \tag{3.2}$$

So, by (3.2), $\{d(x_n, x_{n-1})\}$ is a decreasing sequence of non-negative real numbers, and hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

In the sequel, we prove that $r = 0$.

Assume $r > 0$, then from (3.2) we have

$$0 < \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(d(x_{n-1}, x_n)) < 1, \tag{3.3}$$

for any $n \in \mathbb{N}$. By the Sandwich theorem, from the inequality 3.3, we get $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1$, which contradicts with the continuity of $\beta \in \Gamma$. Hence we obtain $r = 0$. Therefore, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, which means $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$. Now, from the inequality

$$d(f(x_n), f(x_m)) \leq \frac{1}{2}\beta(d(x_n, x_m))[d(x_n, f(x_n)) + d(x_m, f(x_m))]$$

for all $m, n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_{m+1}) \leq \frac{1}{2}\beta(d(x_n, x_m))[dx_n, x_{n+1}) + d(x_m, x_{m+1})],$$

which implies that both sequences $\{x_n\}$ or $\{f(x_n)\}$ are Cauchy sequences. Since (X, d) is complete, the sequence $\{f(x_n)\}$ is convergent to a point u, and also, $x_n \to u$. Indeed, u is the fixed point of f, because we have:

$d(u, f(u)) = \lim_{n\to\infty} d(f(x_n), f(u))$
$\leq \lim_{n\to\infty} \frac{1}{2}\beta(d(x_n, u))[d(u, f(u) + d(x_n, fx_n)]$
$= \frac{1}{2}(\lim_{n\to\infty} \beta(d(x_n, u)))[d(u, f(u)) + 0]$
$\leq \frac{1}{2}d(u, f(u))$,

which gives $d(u, f(u)) = 0$, and then $f(u) = u$. Finally, we show that f cannot to have another fixed point. Assuming a point z to be a fixed point of f, we have

$$d(u, z) = d(f(u), f(z)) \leq \beta(d(u, z))[d(u, f(u)) + d(z, f(z))] = 0,$$

hence $z = u$.

Theorem 3.2. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a Kannan-Geraghty mapping defined as Definition 2.8. Suppose that $T(A_0) \subseteq B_0$ and the pair (A, B) has the weak P-property. Then T has a unique best proximity point $x^\star$ in A such that $d(x^\star, Tx^\star) = d(A, B)$.*

Proof. We first prove that B_0 is closed. Let $\{y_0\} \subseteq B_0$ be a sequence such that $y_n \to q \in B$. It follows from the weak $P-$property that

$$d(y_n, y_m) \to 0 \Rightarrow d(x_n, x_m) \to 0$$

as $n, m \to \infty$, where $x_n, x_m \in A_0$ and $d(x_n, y_n) = d(A, B), d(x_m, y_m) = d(A, B)$. Then $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p \in A$. By the continuity of metric d we have $d(p, q) = d(A, B)$, that is, $q \in B_0$ and hence B_0 is closed.

Let $\overline{A_0}$ be the closure of A_0. We claim that $T(\overline{A_0}) \subseteq B_0$. In fact, if $x \in \overline{A_0}$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. By the continuity of T and the closedness of B_0 we have $Tx = \lim_{n\to\infty} Tx \in B_0$. That is $T(\overline{A_0}) \subseteq B_0$.

Define an operator $PA_0 : T(\overline{A_0}) \to A_0$, by $PA_0 = \{x \in A_0 : d(x, y) = d(A, B)\}$. Since the pair (A, B) has the weak $P-$property

$$
\begin{aligned}
d(PA_0Tx_1, PA_0Tx_2) &\leq d(Tx_1, Tx_2) \\
&\leq \frac{\beta(d(x,y))}{2}[d(x_1Tx_1) + d(x_2, Tx_2) - d(A, B) \\
&\leq \frac{\beta(d(x,y))}{2}[d(x_1, PA_0Tx_1) + d(x_1, PA_0Tx_1) \\
&+ d(x_2, PA_0Tx_2) + d(x_2, PA_0Tx_2) - 2d(A, B) \\
&\leq \frac{\beta(d(x,y))}{2}[d(x_1, PA_0Tx_1) + d(x_2, PA_0Tx_2)].
\end{aligned}
$$

For any $x_1, x_2 \in \overline{A_0}$. This shows that $PA_0T : \overline{A_0} \to \overline{A_0}$ is a Kannan Graghty mapping from complete metric subspace $\overline{A_0}$ into itself. Using Theorem 3.1, we can see that PA_0T a unique fixed point x^*. That is, $PA_0Tx^* = x^* \in A_0$, which implies that

$$d(x^*, Tx^*) = d(A, B).$$

Therefore, x^* is the unique one in A_0 such that $d(x^*, Tx^*) = d(A, B)$. It is easy to see that x^* is also the unique one in A such that $d(x^*, Tx^*) = d(A, B)$. The Picard iteration sequence

$$x_{n+1} = PA_0Tx_n, n = 0, 1, 2, ...$$

converges, for every $x_0 \in A_0$, to x^*. Since the iteration sequence $\{x_{2k}\}_{n=0}^{\infty}$ defined by

$$x_{2k+1} = Tx_{2k},\ d(x_{2k+1}, x_{2k+2}) = d(A, B),\ k = 0, 1, 2...,$$

is exactly the subsequence of $\{x_n\}$, so it converges, for every $x_0 \in A_0$, to x^*. This completes the proof.

Now we proof the following theorem which shows the relation between Kannan-Geraghty and weakly Geraghty.

Theorem 3.3. *Let (X,d) be a complete metric space. If $f : X \to X$ satisfies the following conditions such that*

(1) *Let $f : X \to X$ is weakly Kannan mapping, then f has a unique fixed point x^*;*

(2) *Let $f : X \to X$ is Kannan-Geraghty mapping, then f has a unique fixed point x^*;*

We have if 1 *then* 2.

Proof. $1 \to 2$

A mapping $f : X \to X$ is said to be weakly Kannan provided that

$$d(f(x),(y)) \leq \frac{\overline{\alpha}}{2}(x,y)[d(x,fx)+f(y,fy)]$$

for all $x,y \in X$, where the function $\overline{\alpha} : X \times X \to [0,1)$, for every $0 < a \leq b$, satisfy

$$\Theta(a,b) = \sup\{\overline{\alpha}(x,y) : a \leq d(x,y) \leq b\} < 1.$$

Put $\overline{\alpha}(x,y) = \beta(d(x,y))$, suppose $L = \sup\{\alpha(x,y) \mid a \leq d(x,y) \leq b\} < 1$, for $0 < a < b$. Let $\sup\{L\} \to 1$ then, $\alpha(x,y) \to 1$ or $\beta(d(x_n,y_n)) \to 1$ therefore $d(x_n,y_n) \to 0$, this is a contradiction . Because $0 < a < d(x_n,y_n)$. Hence $a = b = 0$, then f is Kannan Graghty and

$$d(f(x),(y)) \leq \beta(d(x,y))[d(x,fx)+f(y,fy)].$$

By using theorem 2.1 f has a unique fixed point.

Example 3.1. *Let $X = \{(1,x) : 0 \leq x \leq \frac{1}{10}\}$, and define $f : X \to X$ as follows:*

$$f(1,y) = (1,\frac{y^2}{y+1}).$$

We have

$$\begin{aligned}
|\frac{y_1^2}{y_1+1} - \frac{y_2^2}{y_2+1}| &\leq |y_2^2 - y_2^2| \\
&\leq |y_1 - y_2|\ |y_1 + y_2| \\
&\leq \frac{1}{5}|y_1 - y_2| \\
&\leq \frac{1}{5}|y_1 - \frac{y_1^2}{y_1+1} + \frac{y_1^2}{y_1+1} - (y_2 - \frac{y_2^2}{y_2+1} + \frac{y_2^2}{y_2+1})| \\
&\leq \frac{1}{5}|y_1 - \frac{y_1^2}{y_1+1}| + |\frac{y_1^2}{y_1+1} - \frac{y_2^2}{y_2+1}| + |y_2 - \frac{y_2^2}{y_2+1}|
\end{aligned}$$

Then

$$\frac{4}{5}|\frac{y_1^2}{y_1+1} - \frac{y_2^2}{y_2+1}| \leq \frac{1}{5}(|y_1 - \frac{y_1^2}{y_1+1}| + |y_2 - \frac{y_2^2}{y_2+1}|),$$

now we get result

$$\begin{aligned}|\frac{y_1^2}{y_1+1} - \frac{y_2^2}{y_2+1}| &\leq \frac{1}{4}(|y_1 - \frac{y_1^2}{y_1+1}| + |y_2 - \frac{y_2^2}{y_2+1}|)\\ &\leq \frac{1}{2}.\frac{1}{1+d(y_1-y_2)}(|y_1 - \frac{y_1^2}{y_1+1}| + |\frac{y_2^2}{y_2+1}|)\end{aligned}$$

then f is weakly Kannan. with

$$\beta(d(y_1, y_2)) = \frac{1}{1+d(y_1-y_2)}.$$

Then f is weakly Graghty. Put $\frac{1}{1+d(y_1-y_2)} = \overline{\beta}(x,y)$. So for every $0 < a \leq b$. By using theorem 3.3, f is Kannan Graghty and has a unique fixed point.

Theorem 3.4. *Let X be a compact metric space and $f : X \to X$ be a Kannan mapping, and for arbitrary $x_0 \in X$, the Picard iteration process defined by $x_n = f(x_{n-1})$ for $n > 0$. Then f has a unique fixed point x_∞ in X, $x_n \to x_\infty$ in X, iff there exists a subsequence x_{h_n} and x_{k_n} ($x_{h_n} \neq x_{k_n}$) such that*

$$\Delta_n \to 1 \text{ only if } d_n \to 0.$$

Proof. There exists a subsequence x_{h_n} and x_{k_n}, such that $x_n \to x_\infty$. Then clearly $d_n = d(x_{h_n}, x_{k_n}) \to 0$, and the condition holds.

Next, for given initial point x_0 in X, we assume that the condition is satisfied. then $d_n = d(x_n, x_{n+1})$ is non-increasing, for $d(x_{h_n}, x_{k_n}) \leq 1$, and then it is convergent to the real number d, such that $d \to \varepsilon (0 \leq \varepsilon)$. Assume $\epsilon > 0$, and $h_n = n$ and $k_n = n+1$, so we have $d_n \to \epsilon > 0$, While $\Delta \to 1$, which is a contradiction. Thus $d(x_n, x_{n+1}) \to 0$.

Suppose, for a contradiction, that the sequence of iterates $\{x_n\}$ is not Cauchy, the real number

$$D_N = \sup_{m,n \geq N} d(x_n, x_m) > \varepsilon.$$

is called the diameter of the sequence $\{x_n\}_{n \geq N}$, so there exists $\varepsilon > 0$ such that $D_N > \varepsilon$. For any $n > 0$, We choose N_n sufficiently large number, such that $d(x_m, x_{m+1}) < \frac{1}{n}$ for all $m > N_m$, Let h_n is the smallest integer such that $h_n \geq N_n$. For $k_n > h_n$, we have $d(x_{h_n}, x_{k_n}) > \varepsilon$. Such pairs exist by the above diameter condition.

Again we consider the sequence k_n, and put $k_n - 1 = h_n$ or else $d(x_{h_n}, x_{k_n-1}) \leq \varepsilon$. In either case we have $\varepsilon \leq d_n = d(x_{h_n}, x_{k_n}) < \varepsilon + 1$.

Moreover, by using the triangular inequality, for all $x_{h_n}, x_{k_n} \in X$, we have

$$\begin{aligned}
d(x_{h_n+1}, x_{h_n+2}) &= d(f(x_{h_n}), f(x_{h_n+1})) \\
&= d(f(x_{h_n}), f(x_{k_n})) \\
&\leq \frac{\beta}{2}(d(x_{h_n}, x_{h_n+1}) + d(x_{h_n+1}, x_{h_n+2})) \\
&= d(f(x_{h_n}), f(x_{h_n+1})).
\end{aligned}$$

Where $\beta \in [0, 1)$. So

$$d(x_{h_n+1}, x_{h_n+2}) \leq \beta d(x_{h_n}, x_{h_n+1}).$$

Without loss of generality, we may assume that $\dfrac{d(x_{h_n+1}, x_{h_n+2})}{d(x_{h_n}, x_{h_n+1})} > 1 - \dfrac{1}{n}$, so

$$1 \geq \Delta_n = \frac{d(x_{h_n+1}, x_{h_n+2})}{d(x_{h_n}, x_{h_n+1})} > 1 - \frac{1}{n}.$$

So $d_n \to \varepsilon > 0$ while $\Delta_n \to 1$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X and X is complete, we have $x_n \to x_\infty$ for some x_∞ in X, then x_∞ is a unique fixed point of f and the proof is complete.

we present a theorem and after that we bring S$-$Kannan theorem which shows the relation between Kannan and contractive mapping. The proof of our main theorem is inspired by this theorem.

Theorem 3.5. *[S$-$Kannan] Let X be a compact metric space and $f : X \to X$ be a Kannan mapping, and let for arbitrary $x_0 \in X$ the Picard iteration process defined by $x_n = f(x_{n-1})$ for $n > 0$. Then f has a unique fixed point x_∞ in X, $x_n \to x_\infty$ in X, iff*
(i) there exists $\beta : X \times X \to [0, 1)$, such that for every $0 < a \leq b$ and for all n, m and $x_n, x_m \in X$

$$\beta(x_n, x_m) = \sup \{ \ \alpha(x_n, x_m) : a \leq d(x_n, x_m) \leq b\} < 1$$

and

$$d(f(x_n), f(x_m)) \leq \frac{\beta(x_n, x_m)}{2}(d(x_n, f(x_n)) + d(x_m, f(x_m))).$$

(ii) $\beta(x_n, x_m) \in S$

Proof. It suffices to prove that β in S satisfies the condition of Theorem 2.2. Let $\beta = \gamma(d(x_n, y_n))$, where the class γ denotes function $\gamma : [0, \infty) \to [0, 1)$ satisfying the following condition

$$\gamma(t_n) \to 1 \Rightarrow t_n \to 0$$

for every $0 < a < b$, let $L = \sup\{\alpha(x_n, y_n) | a \leq d(x_n, y_n) \leq b\}$, and let $\limsup\{L\} = 1$ implies $\beta(x_n, y_n) \to 1$ or $\gamma(d(x_n, y_n)) \to 1$. Hence $d(x_n, y_n) \to 0$. it is a contradiction, because $0 < a < d(x_n, y_n)$, so $a = b = 0$. So γ holds in delta, the conclusion is exactly the same as what we had in theorem 3.4.

Define $\beta : \mathbb{R}^+ \to \mathbb{R}$ by

$$\beta = \sup\{\frac{2d(f(x_n), f(x_m))}{d(x_n, f(x_n) + d(x_m, f(x_m))} = d(x_n, x_m) \geq t\} = \alpha(d(x_n, x_m)) = \alpha(t_n).$$

Since f is a Kannan , all quotients are below 1 and so β is defined for all $t > 0$ and $\beta \leq 1$. It is clear that β satisfies in (i).

Before presenting an important result, we first present a preliminary result:

let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_n = fx_{n_1}, n = 1, 2, ...,$ we have

$$d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) \leq \frac{\beta}{2}\ (d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)),$$

so

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{2}\ \frac{\beta}{1-\beta}\ d(x_n, x_{n+1}),$$

by the same argument,

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{2}\ (\frac{\beta}{1-\beta})^n\ d(x_1, x_0). \tag{3.4}$$

By (3.4), for every $m, n \in \mathbb{N}$ such that $n > m$ we have

$$d(f(x_n), f(x_m)) \leq \frac{\beta}{2}(d(x_n, f(x_n)) + d(x_m, f(x_m))),$$

or

$$d(x_{n+1}, x_{m+1}) \leq \frac{\beta}{2}(d(x_n, f(x_n)) + d(x_m, f(x_m))).$$

So

$$\begin{aligned} d(x_{n+1}, x_{m+1}) &\leq \frac{\beta}{2}(d(x_n, x_{n+1}) + d(x_m, x_{m+1})) \\ &\leq \frac{\beta}{2}(d(x_n, x_m) + d(x_{m+1}, x_{n+1})) + \beta d(x_m, x_{m+1}) \\ &\leq \frac{\beta}{2}(d(x_n, x_m) + d(x_{m+1}, x_{n+1})) + (\frac{\beta}{1-\beta})^m d(x_0, x_1). \end{aligned}$$

So

$$d(x_{n+1}, x_{m+1}) \leq \beta \; d(x_n, x_m),$$

or

$$d(f(x_n), f(x_m)) \leq \beta \; d(x_n, x_m).$$

Now, let $\beta(t_n) \to 1$ for $t_n \in \mathbb{R}^+$. We may further assume without loss of generality that

$$1 - \frac{1}{n} < \frac{d(f(x_n), f(x_m))}{d(x_n, x_m)} \leq \beta = \alpha(t_n) \leq 1.$$

We must show that $t_n \to 0$. Since $\alpha(t_n)$ is an upper bound. So for each $n > 0$, there are x_{h_n} and x_{k_m} in $\{x_n\}$, such that

$$d(x_{h_n}, x_{h_m}) \geq t_n,$$

and

$$1 - \frac{1}{n} < \frac{d(fx_{k_n}), f(x_{k_m})}{d(x_{k_n}, x_{k_m})} \leq \beta = \alpha(t_n) \leq 1.$$

So $\Delta_n \to 1$. Hence from theorem 3.4, we have $d(x_{h_n}, x_{k_n}) \to 0$. So $t_n \to 0$. This completes the proof.

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REMARKS ON THE GEOMETRY AND THE TOPOLOGY OF THE LOOP SPACES $H^s(S^1, N)$, FOR $s \leq 1/2$.

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Abstract. We first show that, for a fixed locally compact manifold N, the space $L^2(S^1, N)$ has not the homotopy type of the classical loop space $C^\infty(S^1, N)$, by two theorems:

- the inclusion $C^\infty(S^1, N) \subset L^2(S^1, N)$ is null homotopic if N is connected,
- the space $L^2(S^1, N)$ is contractible if N is compact and connected.

Then, we show that the spaces $H^s(S^1, N)$ carry a natural structure of Frölicher space, equipped with a Riemannian metric, which motivates the definition of Riemannian diffeological space.

1. Introduction

The objects studied in this paper are spaces of maps from $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ to a locally compact, connected manifold N embedded into an Euclidean space V, usually called loop spaces. The most studied loop space is the space of smooth loops $C^\infty(S^1, N)$, or of smooth based loops $C_b^\infty(S^1, N)$ where only loops starting and ending at a fixed basepoint $x_0 \in N$ are considered. Embedding $C^\infty(S^1, N)$, resp. $C_0^\infty(S^1, N)$, into $C^\infty(S^1, V)$, one can consider spaces of loops with lower regularity in the following way: considering e.g. the Sobolev

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spaces $H^s(S^1,V)$, a simple way to define such loop spaces is to make the closure of $C^\infty(S^1,N)$, resp. $C_0^\infty(S^1,N)$, in $H^s(S^1,V)$, noted $H^s(S^1,N)$, resp. $H_0^s(S^1,N)$. For $s=0$, we get the space of loops in the L^2-class, noted by $L^2(S^1,N)$, and when the target space is a Lie group G, $H^s(S^1,G)$, resp. $H_b^s(S^1,G)$, is called the H^s-loop group, resp. the H^s-based loop group. The loop spaces $H^s(S^1,N)$, for $s>1/2$ are well-known Hilbert manifolds, and the loop groups $H^s(S^1,G)$ are Hilbert Lie groups, since [14]. But

very often geometry and analysis stops for $s \leq 1/2$, because the classical construction of a smooth atlas on these spaces requires an inclusion into the space of continuous loops $C^0(S^1,N)$, via Sobolev embedding theorems. The same holds for loop groups $H^s(S^1,G)$, see e.g. [27], where one can read also that, for $s=1/2$, most loops $\gamma \in H^{1/2}(S^1,G) - C^0(S^1,G)$ are not easy to study. One can extend this remark to $s \leq 1/2$.

The aim of this paper is to give a first approach of some topological properties of some of these spaces for $s \leq 1/2$, and propose an adapted geometric setting. In a first part of the paper (section 3), we show that there is no homotopy equivalence between $L^2(S^1,N)$ and $H^s(S^1,N)$ for $s>1/2$, which furnishes a great contrast with the known: for $s>k>1/2$, the inclusion $H^s(S^1,N) \subset H^k(S^1,N)$ is a homotopy equivalence [26, 14, 9].

Motivated by the fact that mathematical literature often use weak Sobolev metrics on $C^\infty(S^1,N)$, especially $H^{1/2}$ and L^2-metrics (see e.g.[16, 27, 30]), a natural question is the geometric setting that would enable to discuss with the topologico-geometric properties of the full spaces $H^s(S^1,N)$ for $s \leq 1/2$. We then need to find a setting that describes finer structures than the topology, and which enables techniques of differential geometry. We choose here the setting of Frölicher spaces, which can be seen as a particular case of diffeological space [22, 3, 29], and we develop for the needs of the example of loop spaces the notion of Riemannian diffeological space. As a final remark (section 5.2), we show that the canonical (weak, $H^{1/2}$) symplectic form on the based loop space naturally extends to the (full) based loop $H_0^{1/2}(S^1,N)$, while the Kähler form of the based loop group does not have the same properties.

2. Preliminaries on loop groups and loop spaces

Let $I=[0;1]$. We note by $(f_n)_{n\in\mathbb{Z}}$ the Fourier coefficients of any smooth map f. Recall that, for $s \in \mathbb{R}$, the space $H^s(I,\mathbb{C})$ is the completion of $C^\infty(I,\mathbb{C})$ for the norm $||.||_s$ defined

by

$$||f||_s^2 = \sum_{n\in\mathbb{Z}}(1+|n|)^{2s}|f_n|^2 = \int_{S^1}(1+\Delta^{1/2})^{2s}(f).\bar{f},$$

where $\Delta = -\frac{1}{4\pi^2}\frac{d^2}{dx^2}$ is the standard Laplacian, and $S^1 = \mathbf{R}/\mathbf{Z}$. The same construction holds replacing $\mathbb{C}$ by an algebra $\mathcal{M}$ of matrices with complex coefficients, with the hermitian product of matrices $(A,B)\mapsto tr(AB^*)$. If there is no possible confusion, we note this matrix norm by $||.||$ or by $||.||_{\mathcal{M}}$ if necessary. Let N be a smooth connected manifold, with Riemannian embedding into $\mathcal{M}$. We can assume that the $0-$matrix, noted 0, is in N with no loss of generality since the space of Riemannian embeddings from N to $\mathcal{M}$ is translation invariant in $\mathcal{M}$. The loop space $C^\infty(S^1,N)$ is a smooth Fréchet manifold (see [14, 9] for details). The submanifold of based loops $C^\infty(S^1,N)$ is here identified with loops $\gamma\in C^\infty(S^1,N)$ such that $\gamma(0)=0$. Let us now consider a compact connected Lie group G of matrices. We note by $C^\infty(S^1,G)$, resp. $C_0^\infty(S^1,G)$, the group of smooth loops, resp. the group of based smooth loops γ such that $\gamma(0)=\gamma(1)=e_G$. (When dealing with based loop groups, the chosen basepoint is the identity matrix Id for trivial necessities of compatibility with the group multiplication)

Definition 2.1. *We define $H^s(S^1,G)$, resp. $H_0^s(S^1,G)$, as the adherence of $C^\infty(S^1,G)$, resp. $C_0^\infty(S^1,G)$, in $H^s(S^1;\mathcal{M})$.*

For $s>1/2$, it is well-known, that $H^s(S^1,G)$ is a Hilbert Lie group. The key tool is the smooth inclusion $H^s(S^1,\mathcal{M})\subset C^0(S^1,\mathcal{M})$, which enables to define charts via tubular neighborhoods, and to define the group multiplication and the group inversion pointwise by the smoothness of the evaluation maps, see the historical paper [14] for details, see also [27] for an exposition centered on loop groups. The biggest Sobolev order where this fails is $s=1/2$. For $s>1/2$,

(1) the norm $||.||_s$ induces a (strong) scalar product $<.,.>_s$ on $H^s(S^1,\mathfrak{g})$, which induces a left invariant metric on $TH_s(S^1,G)$.
(2) if $1/2<s$, the H^k-scalar product $<.,.>_k$ induces a weak Riemannian metric on $TH_s(S^1,G)$, but the H^k-geodesic distance is non vanishing on $H^s(S^1,G)$, for $k<s$.

The motivation of this last remark can be found in recent works [6, 7, 8, 24] where are given some examples of weak Sobolev H^s metrics on manifolds of mappings with vanishing geodesic distance.

3. The case $s = 0$: On the homotopy type of $L^2(S^1,N)$

Let us now analyze $L^2(S^1,N)$ when N is connected.

Lemma 3.1. *Let $T = \{(l,s) \in I^2 | s \leq l \text{ and } l > 0$. There exists a map $\varphi \in C^\infty(T,[0;1])$ such that*

$$\begin{cases} \forall l, \varphi(l,0) = 0 \\ \forall l, \varphi(l,l) = 1 \\ \forall l, \frac{\partial \varphi}{\partial s}(l,0) = 1 \\ \forall l, \frac{\partial \varphi}{\partial s}(l,l) = 1 \\ \forall l, \forall k > 1, \frac{\partial^k \varphi}{\partial s^k}(l,0) = 0 \\ \forall l, \forall k > 1, \frac{\partial^k \varphi}{\partial s^k}(l,l) = 0 \end{cases}.$$

One can choose the following map:

$$\varphi(l,s) = \int_0^s (\phi_l * m_{1/6l})(t)dt$$

where

$$\begin{aligned} \phi_l : \quad \mathbf{R} &\to \mathbf{R} \\ t &\mapsto \begin{cases} 1 & \text{if } t < l/3 \text{ or } t > 2l/3 \\ \frac{(3-2l)}{l} & \text{otherwise} \end{cases} \end{aligned}$$

and $m_{1/6l}(t) = 6lm(t/6l)$, with m a standard mollifier with $[-1;1]$ support.

Proof. The solution given fulfills the conditions required by classical results of analysis.

Notice that with such a function φ, we have that $\frac{\partial \varphi}{\partial s}(l,s) \leq 3/l$, and that $\frac{\partial \varphi}{\partial l}(l,s) \leq M(s) \in \mathbf{R}_+^*$, for any fixed $0 < s \leq 1$. For a sequence (s_n) such that $s_n \to 0$, the sequence $M(s_n)$ is unbounded, were as one can take $M(0) = 1$. These properties are necessary for the proofs of the rest of the section.

Theorem 3.1. *$L^2_0(S^1,N) = L^2(S^1,N)$ and for any loop γ in $C^\infty(S^1,N)$ there is a map $P \in C^0([0;1], L^2(S^1,N))$ such that $P(0) = \gamma$ and $P(1)$ is null-homotopic piecewise smooth loop in L^2.*

Proof. We now assume that $0 \in N$. Let $\gamma \in C^\infty(S^1,N)$ such that $\gamma(0) = x \neq 0$. Let $\tau : [0;1] \to N$ be a null-homotopic smooth loop such that $\tau(0) = Id$, $\tau(1/2) = x$, $\tau(1) = 0$, $\dot\tau(1/2) = \dot\gamma(0)$ in T_xN. Such a loop exists considering a neighborhood in N of a smooth path

starting from 0 and finishing at x. Let $T = \{(l,s) \in I^2 | s \leq l$ and $l > 0$. Let $\varphi \in C^\infty(T, [0;1])$ such that

$$\begin{cases} \forall l, \varphi(l,0) = 0 \\ \forall l, \varphi(l,l) = 1 \\ \forall l, \frac{\partial \varphi}{\partial s}(l,0) = 1 \\ \forall l, \frac{\partial \varphi}{\partial s}(l,l) = 1 \\ \forall l, \forall k > 1, \frac{\partial^k \varphi}{\partial s^k}(l,0) = 0 \\ \forall l, \forall k > 1, \frac{\partial^k \varphi}{\partial s^k}(l,l) = 0 \end{cases}$$

With the example given in Lemma 3.1, we have that

$$max_{s \in [0;1]} \frac{\partial \varphi}{\partial s}(l,s) \leq 3/l.$$

Let us consider the family of piecewise smooth paths $h \in [0;1] \mapsto \gamma_h$ such that

$$\gamma_h(s) = \begin{cases} \tau \circ \varphi(h/2, s) & \text{if} & s \leq h/2 \\ \gamma \circ \varphi(1-h, s-h/2) & \text{if} & h/2 < s < 1-h/2 \\ \tau \circ \varphi(h/2, 1-s) & \text{if} & s \geq 1-h/2 \end{cases}.$$

One can check that this is in fact a smooth path, considering the Taylor series at the connecting points $s = h/2$, $s = 1-h/2$ and $s = 0$. Let us take the limit when $h \to 0$.

$$\begin{aligned} ||\gamma_h - \gamma||^2_{L^2(S^1,\mathcal{M}} &= \int_0^{h/2} ||\gamma_h(s) - \gamma(s)||^2_{\mathcal{M}} ds + \\ & \int_{h/2}^{1-h/2} ||\gamma_h(s) - \gamma(s)||^2_{\mathcal{M}} ds + \\ & \int_{1-h/2}^{h/2} ||\gamma_h(s) - \gamma(s)||^2_{\mathcal{M}} ds \\ &= \int_0^{h/2} ||\tau \circ \varphi(h/2, s) - \gamma(s)||^2_{\mathcal{M}} ds + \\ & \int_{h/2}^{1-h/2} ||\gamma \circ \varphi(1-h, s-h/2) - \gamma(s)||^2_{\mathcal{M}} ds + \\ & \int_{1-h/2}^{h/2} ||\tau \circ \varphi(h/2, 1-s) - \gamma(s)||^2_{\mathcal{M}} ds \\ &\leq \frac{h(sup_{s\in[0;1]}||\tau(s)|| + sup_{s\in[0;1]}||\gamma(s)||)^2}{2} + \\ & (1-h). \left(hM(1-h) sup_{s\in[0;1]}||\dot\gamma(s)|| + \frac{3h sup_{s\in[0;1]}||\dot\gamma(s)||}{2(1-h)} \right)^2 + \\ & \frac{h(sup_{s\in[0;1]}||\tau(s)|| + sup_{s\in[0;1]}||\gamma(s)||)^2}{2} \end{aligned}$$

So that,

$$\lim_{h\to 0}\gamma_h=\gamma,$$

which shows that γ is in the L^2- closure of $C_0^\infty(S^1,N)$. Thus we get that $L^2(S^1,N)=L_0^2(S^1,N)$. On the other hand, when we take the L^2-limit of γ_h when $h\to 1$, we get with the same techniques:

$$\lim_{h\to 1}\gamma_h=\tau\vee\tau^{-1}.$$

Let us now give another result from the techniques described in the proof of last theorem:

Theorem 3.2. *The natural injection $C_0^\infty(S^1,N)\to L^2(S^1,N)$ is homotopic to a constant map.*

Proof. Let $\gamma\in C_0^\infty(S^1,\mathcal{M})$. Let

$$H(s',\gamma)(l)=\gamma\circ\varphi(1-s';l)\text{ for }0\leq l\leq 1-s',$$

with φ defined by Lemma 3.1 that we extend by the constant path on $[1-s;1]$. $H(0,\gamma)=\gamma$ and $H(1,\gamma)$ is the constant loop. For $0<s'<1$ $H(s,\gamma)$ is a piecewise smooth loop, with only one angular point at $l=1-s'$, and hence is in L^2. We have now to get a continuity property for the map $s'\mapsto H(s',\gamma)$. Let s,t such that $0\leq s<t\leq 1$ with $t-s<\frac{s}{6}<\frac{t}{6}$. We are using in the sequel the following majorations:

- the classical estimate of the Lipschitz constant of a C^1 path: $||\gamma(s)-\gamma(t)||\leq(t-s)max_{s\leq l\leq t}||\dot\gamma(l)||$,
- and since S^1 is compact, $max_{s\in[0;1]}||\gamma(s)||=k_\gamma\leq+\infty$

which implies

- on the one hand, since $t-s<\frac{s}{6}<\frac{t}{6}$, for $l\in[1-t;1-s]$, $\frac{\partial\varphi}{\partial l}(1-s,l)=1$, which implies:

$$\begin{aligned}\int_{1-t}^{1-s}||\gamma\circ\varphi(1-s;l)-\gamma(1)||^2dl &\leq max_{0<l<1}||\dot\gamma(l)||^2\int_{1-t}^{1-s}|\varphi(1-s,l)-\varphi(1-s,1-s)|^2dl\\ &\leq (t-s)max_{0<l<1}||\dot\gamma(l)||^2\end{aligned}$$

- and on the other hand, setting $M(l)$ the Lipschitz constant of the map $\phi(.,l)$, :

$$|\varphi(1-s,l)-\varphi(1-t,l)|\leq M(l)(t-s)\Rightarrow||\gamma\circ\varphi(1-s,l)-\gamma\circ\varphi(1-t,s)||\leq M(l)(t-s)max_{0<l<1}||\dot\gamma(l)||.$$

On the interval $[(1-t),(1-s)]\subset[0;1[$, we have that $M(l)$ is bounded by a constant noted $k_{s,t}$.

With these two inequalities, we get:

$$\begin{aligned}
||H(s,\gamma)-H(t,\gamma)||^2_{L^2(S^1,\mathcal{M})} &= \int_0^{1-t} ||\gamma\circ\varphi(1-s;l)-\gamma\circ\varphi(1-t;l)||^2 dl + \\
&\int_{1-t}^{1-s} ||\gamma\circ\varphi(1-s;l)-\gamma(1)||^2 dl \\
&+\int_{1-s}^{1} ||\gamma(1)-\gamma(1)||^2 dl \\
&\leq k_{s,t}(1-t)(t-s)max_{0<l<1}||\dot\gamma(l)||^2 \\
&+(t-s)max_{0<l<1}||\dot\gamma(l)||^2+0
\end{aligned}$$

Hence, for a fixed smooth loop γ, $s'\mapsto H(s',\gamma)\in C^0(]0;1];L^2(S^1,\mathcal{M}))$. We need to show continuity at $s'=0$. We get the following inequalities:

$$\begin{aligned}
||H(0,\gamma)-H(t,\gamma)||^2_{L^2(S^1,\mathcal{M})} &= \int_{1-t}^{1} ||\gamma\circ\varphi(1-t;l)-\gamma(1)||^2 dl \\
&\leq t max_{0<l<1}||\dot\gamma(l)||^2
\end{aligned}$$

which completes the continuity in the first parameter. We remark that, for fixed γ, H is not Lipschitz in the first parameter, since $k_{s,t}$ is not bounded for (s,t) in the neighborhood of $(0;0)$.

We now have to show that the map $\gamma\mapsto H(s,\gamma)$ is continuous for the L^2-topology. For this, we only have to remark the change of coordinates formula:

$$||\gamma-\tau||^2_{L^2(S^1,\mathcal{M})}=\int_0^{1-s}||\gamma\circ\varphi(1-s,l)-\tau\circ\varphi(1-s,l)||^2\partial_l\varphi(1-s,l)dl.$$

Since $\partial_l\varphi(1-s,l)>1$ for $0\leq l\leq 1-s$, we get

$$\begin{aligned}
||H(s,\gamma)-H(s,\tau)||^2_{L^2(S^1,\mathcal{M})} &= \int_0^{1} ||\gamma\circ\varphi(1-s,l)-\tau\circ\varphi(1-s,l)||^2 dl \\
&= \int_0^{1-s} ||\gamma\circ\varphi(1-s,l)-\tau\circ\varphi(1-s,l)||^2 dl \\
&\leq \int_0^{1-s} ||\gamma\circ\varphi(1-s,l)-\tau\circ\varphi(1-s,l)||^2 \partial_l\varphi(1-s,l)dl \\
&\leq ||\gamma-\tau||^2_{L^2(S^1,\mathcal{M})}
\end{aligned}$$

Hence the map $\gamma\mapsto H(s,\gamma)$ is $1-$Lipschitz.

Corollary 3.1. *Let $k>1/2$. The canonical inclusion $i:H^k_0(S^1,N)\rightarrow L^2(S^1,N)$ induces a $0-$map $i_*:H_*(H^k_0(S^1,N))\rightarrow H_*(L^2(S^1,N))$ and $i_*:\pi_*(H^k_0(S^1,N))\rightarrow\pi_*(L^2(S^1,N))$.*

Proof. If $k > 1/2$, the canonical inclusion $C_0^\infty(S^1, N) \to H_0^k(S^1, N)$ is a well-known homotopy equivalence between smooth manifolds. So that, by Theorem 3.2, we get the result.

We now finish this section with the following result:

Theorem 3.3. *Assume that N is connected and compact. Then the space $L^2(S^1, N)$ is contractible.*

Proof. For convenience of the proof, we assume that $0 \in N$. Let $H(t, \gamma)(s) = 1_{s<t}\gamma(s)$.

- $||H(t, \gamma)||_{L^2(S^1,\mathcal{M})} \leq ||1_{s<t}||_{L^\infty(S^1,\mathcal{M})}||\gamma||_{L^2(S^1,\mathcal{M})} = ||\gamma||_{L^2(S^1,\mathcal{M})}$ so that $H(t, \gamma) \in L^2(S^1, \mathcal{M})$. Remarking that H is linear in the second variable, we get that $H(t, .)$ is smooth on $L^2(S^1, \mathcal{M})$.
- Let $\gamma \in L^2(S^1, N)$. There is a sequence $(\gamma_n)_{n\in\mathbf{N}} \in C_0^\infty(S^1, N)^{\mathbf{N}}$ that converges to γ.

Claim: The sequence $(H(t, \gamma_n))_{n\in\mathbf{N}}$ is in $L^2(S^1, N)$. For this, for fixed t and n, we consider reparametrizations of γ_n for $p \in \mathbf{N}^*$ such that $t - 1/p < 1$:

$$\delta_p(s) = \begin{cases} \gamma_n(s) \text{ if} & s \leq t \\ \gamma_n(t + p(s - t)) \text{ if} & t < s < t + 1/p \\ 0 & \text{otherwise} \end{cases}$$

We have that the sequence (δ_p) is in the Sobolev class H^1, and

$$||\delta_p - H(t, \gamma_n)||_{L^2(S^1,\mathcal{M})} = \left(\int_t^{t+1/p} (\gamma_n(t + p(s - t)))^2 \, ds\right)^{1/2} \leq \frac{||\gamma_n||_{L^2(S^1,\mathcal{M})}}{p}$$

which shows that (δ_p) converges to γ_n. Since $C^\infty(S^1, N)$ is dense in $H^1(S^1, N)$ [14], we get that $(H(t, \gamma_n))_{n\in\mathbf{N}}$ is in $L^2(S^1, N)$.

Now, we have

$$||H(t, \gamma_n) - H(t, \gamma)||_{L^2(S^1,\mathcal{M})} \leq ||\gamma_n - \gamma||_{L^2(S^1,\mathcal{M})}$$

So that $H(t, \gamma) \in L^2(S^1, N)$.

- Let $\gamma \in L^2(S^1, N)$. For $(t', t') \in [0; 1]^2$, with $t' > t$, we get

$$\begin{aligned} ||H(t', \gamma) - H(t, \gamma)||_{L^2(S^1,\mathcal{M})} &\leq ||H(t', \gamma) - H(t', \gamma_n)||_{L^2(S^1,\mathcal{M})} \\ &\quad + ||H(t', \gamma_n) - H(t, \gamma_n)||_{L^2(S^1,\mathcal{M})} \\ &\quad + ||H(t, \gamma_n) - H(t, \gamma)||_{L^2(S^1,\mathcal{M})} \\ &\leq 2||\gamma - \gamma_n||_{L^2(S^1,\mathcal{M})} \\ &\quad + ||H(t', \gamma_n) - H(t, \gamma_n)||_{L^2(S^1,\mathcal{M})} \end{aligned}$$

Now, N is compact so that, $k = sup_{x\in N}||x||_{\mathcal{M}} < +\infty$. For this, we get

$$\begin{aligned} ||H(t',\gamma_n) - H(t,\gamma_n)||_{L^2(S^1,\mathcal{M})} &\leq k||1_{[t,t']}||_{L^2(S^1,\mathcal{M})} \\ &= k(t'-t) \end{aligned}$$

Inserting this last inequality in the previous one, for $\epsilon > 0$, choose $(t'-t) < \epsilon/3k$ and n such that $||\gamma - \gamma_n||_{L^2(S^1,\mathcal{M})} < \epsilon/3$, we get that $||H(t',\gamma) - H(t,\gamma)||_{L^2(S^1,\mathcal{M})} \leq \epsilon$, which shows H is continuous in the first variable, and ends the proof.

Remark 3.1. *The same procedure can be adapted replacing $L^2(S^1,N)$ by $L^2(M,N)$, with M compact manifold. With the same arguments, one can build the homotopy map with a smooth Morse function, and mimick line by line the last proof. This proof will be developped elsewhere, for the sake of unity of the exposition.*

4. Riemannian metrics and Hausdorff measures on diffeological spaces

Diffeological spaces and Frölicher spaces will furnish a setting to deal with the differential geometry of the loop spaces $H^s(S^1,N)$. For preliminaries on diffeological spaces and Frölicher spaces, we refer to [19] and to [17, 20]. For convenience, the necessary material and a complementary bibliography is given in section 4.1. We now describe an extension of some basic structures of Riemannian manifolds to diffeological spaces.

4.1. **Preliminaries on diffeologies and Frölicher spaces.** The objects of the category of -finite or infinite- dimensional smooth manifolds is made of topological spaces M equipped with a collection of charts called maximal atlas that enables one to make differentiable calculus. But in examples of projective limits of manifolds, a differential calculus is needed as no atlas can be defined. To circumvent this problem which occurs in various frameworks, several authors have independently developed some ways to define differentiation without defining charts. We use here two of them. The first one is due to Souriau ([28], see e.g. [19] for a textbook), the second one is due to Frölicher (see [17], and e.g. [20] for an introduction). In this section, we review some basics on these notions. A (non exhaustive) complementary list of reference is [3, 4, 5, 10, 11, 12, 13, 21, 22, 23, 29].

4.1.1. *Diffeological spaces and Frölicher spaces.*

Definition 4.1. *Let X be a set.*

• *A **parametrization** of dimension p (or p-plot) on X is a map from an open subset O of $\mathbf{R}^p$ to X.*

• *A **diffeology** on X is a set $\mathcal{P}$ of parametrizations on X such that, for all $p \in \mathbf{N}$,*

- any constant map $\mathbf{R}^p \to X$ is in $\mathcal{P}$;

- Let I be an arbitrary set; let $\{f_i : O_i \to X\}_{i \in I}$ be a family of maps that extend to a map $f : \bigcup_{i \in I} O_i \to X$. If $\{f_i : O_i \to X\}_{i \in I} \subset \mathcal{P}$, then $f \in \mathcal{P}$.

- (chain rule) Let $f \in \mathcal{P}$, defined on $O \subset \mathbf{R}^p$. Let $q \in \mathbf{N}$, O' an open subset of $\mathbf{R}^q$ and g a smooth map (in the usual sense) from O' to O. Then, $f \circ g \in \mathcal{P}$.

• *the parametrizations $p \in \mathcal{P}$ are called the* **plots** *of the diffeology $\mathcal{P}$.*

• *If $\mathcal{P}$ is a diffeology X, $(X, \mathcal{P})$ is called a **diffeological space**.*

*Let $(X, \mathcal{P})$ et $(X', \mathcal{P}')$ be two diffeological spaces, a map $f : X \to X'$ is **differentiable** (=smooth) if and only if $f \circ \mathcal{P} \subset \mathcal{P}'$.*

Remark 4.1. *Notice that any diffeological space $(X, \mathcal{P})$ can be endowed with a natural topology such that all the maps that are in $\mathcal{P}$ are continuous.This topology is called the $D-$topology* [10].

Remark 4.2. *Let $f \in \mathcal{P}$, defined on $O \subset \mathbf{R}^p$. we call p the dimension of the plot f.*

We now introduce Frölicher spaces.

Definition 4.2. • *A **Frölicher** space is a triple $(X, \mathcal{F}, \mathcal{C})$ such that*

- $\mathcal{C}$ is a set of paths $\mathbf{R} \to X$,

- A function $f : X \to \mathbf{R}$ is in $\mathcal{F}$ if and only if for any $c \in \mathcal{C}$, $f \circ c \in C^\infty(\mathbf{R}, \mathbf{R})$;

*- A path $c : \mathbf{R} \to X$ is in $\mathcal{C}$ (i.e. is a **contour**) if and only if for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbf{R}, \mathbf{R})$.*

• *Let $(X, \mathcal{F}, \mathcal{C})$ et $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces, a map $f : X \to X'$ is **differentiable** (=smooth) if and only if $\mathcal{F}' \circ f \circ \mathcal{C} \in C^\infty(\mathbf{R}, \mathbf{R})$.*

Any family of maps $\mathcal{F}_g$ from X to $\mathbf{R}$ generate a Frölicher structure $(X, \mathcal{F}, \mathcal{C})$, setting [20]:

- $\mathcal{C} = \{c : \mathbf{R} \to X$ such that $\mathcal{F}_g \circ c \subset C^\infty(\mathbf{R}, \mathbf{R})\}$

- $\mathcal{F} = \{f : X \to \mathbf{R}$ such that $f \circ \mathcal{C} \subset C^\infty(\mathbf{R}, \mathbf{R})\}$.

A Frölicher space carries a natural topology, which is the pull-back topology of $\mathbf{R}$ via $\mathcal{F}$, see e.g. [5]. In the case of a finite dimensional differentiable manifold, the underlying topology of the Frölicher structure is the same as the manifold topology. In the infinite dimensional case, these two topologies differ very often.

In the previous settings, we call X a **differentiable space**, omitting the structure considered. Notice that the sets of differentiable maps between two differentiable spaces satisfy the chain rule. Let us now compare these settings: Let $(X, \mathcal{F}, \mathcal{C})$ be a Frölicher space. We define

$$\mathcal{P}(\mathcal{F}) = \coprod_{p \in \mathbf{N}} \{ \, f \text{ p- paramatrization on } X; \, \mathcal{F} \circ f \in C^\infty(O, \mathbf{R}) \quad \text{(in the usual sense)}\}.$$

With this construction, we also get a natural diffeology when $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, extension of the "nébuleuse" diffeology of a manifold [19]. In this case, one can easily show the following:

Proposition 4.1. [22] *Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces. A map $f : X \rightarrow X'$ is smooth in the sense of Frölicher if and only if it is smooth for the underlying diffeologies.*

Thus, we can state in an intuitive but comprehensive way:

smooth manifold $\Rightarrow$ Frölicher space $\Rightarrow$ Diffeological space

4.1.2. *Frölicher completion of a diffeological space.* We now finish the comparison of the notions of diffeological and Frölicher space following mostly [29]:

Theorem 4.1. *Let $(X, \mathcal{P})$ be a diffeological space. There exists a unique Frölicher structure $(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P})$ on X such that for any Frölicher structure $(\mathcal{F}, \mathcal{C})$ on X, these two equivalent conditions are fulfilled:*

(i) the canonical inclusion is smooth in the sense of Frölicher $(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P}) \rightarrow (X, \mathcal{F}, \mathcal{C})$

(ii) the canonical inclusion is smooth in the sense of diffeologies $(X, \mathcal{P}) \rightarrow (X, \mathcal{P}(\mathcal{F}))$.

Moreover, $\mathcal{F}_\mathcal{P}$ is generated by the family

$$\mathcal{F}_0 = \{f : X \rightarrow \mathbf{R} \text{ smooth for the usual diffeology of } \mathbf{R}\}.$$

Proof. Let $(X, \mathcal{F}, \mathcal{C})$ be a Frölicher structure satisfying *(ii)*. Let $p \in P$ of domain O. $\mathcal{F} \circ p \in C^\infty(O, \mathbf{R})$ in the usual sense. Hence, if $(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P})$ is the Frölicher structure on X generated by the set of smooth maps $(X, \mathcal{P}) \rightarrow \mathbf{R}$, we have two smooth inclusions

$$(X, \mathcal{P}) \rightarrow (X, \mathcal{P}(\mathcal{F}_\mathcal{P})) \text{ in the sense of diffeologies}$$

and

$$(X, \mathcal{F}_\mathcal{P}, \mathcal{C}_\mathcal{P}) \rightarrow (X, \mathcal{F}, \mathcal{C}) \text{ in the sense of Frölicher.}$$

Proposition 4.1 ends the proof.

Definition 4.3. [29] *A **reflexive** diffeological space is a diffeological space $(X,\mathcal{P})$ such that $\mathcal{P} = \mathcal{P}(\mathcal{F}_{\mathcal{P}})$.*

Theorem 4.2. [3, 29] *The category of Frölicher spaces is exactly the category of reflexive diffeological spaces.*

This last theorem allows us to make no difference between Frölicher spaces and reflexive diffeological spaces. We shall call them Frölicher spaces, even when working with their underlying diffeologies.

4.1.3. *Push-forward, quotient and trace.* We give here only the results that will be used in the sequel.

Proposition 4.2. [22] *Let $(X,\mathcal{P})$ be a diffeological space, and let X' be a set. Let $f : X \to X'$ be a surjective map. Then, the set*

$$f(\mathcal{P}) = \{u \text{ such that } u \text{ restricts to some maps of the type } f \circ p; p \in \mathcal{P}\}$$

*is a diffeology on X', called the **push-forward diffeology** on X' by f.*

Let $X_0 \subset X$, where X is a Frölicher space or a diffeological space, we can define on trace structure on X_0, induced by X.

• If X is equipped with a diffeology $\mathcal{P}$, we can define a diffeology $\mathcal{P}_0$ on X_0 setting

$$\mathcal{P}_0 = \{p \in \mathcal{P}\text{such that the image of } p \text{ is a subset of } X_0\}.$$

• If $(X,\mathcal{F},\mathcal{C})$ is a Frölicher space, we take as a generating set of maps $\mathcal{F}_g$ on X_0 the restrictions of the maps $f \in \mathcal{F}$. In that case, the contours (resp. the induced diffeology) on X_0 are the contours (resp. the plots) on X which image is a subset of X_0.

4.1.4. *Cartesian products and projective limits.*

Proposition 4.3. *Let $(X,\mathcal{P})$ and $(X',\mathcal{P}')$ be two diffeological spaces. We call **product diffeology** on $X \times X'$ the diffeology $\mathcal{P} \times \mathcal{P}'$ made of plots $g : O \to X \times X'$ that decompose as $g = f \times f'$, where $f : O \to X \in \mathcal{P}$ and $f' : O \to X' \in \mathcal{P}'$.*

In the case of a Frölicher space, we derive very easily, compare with e.g. [20]:

Proposition 4.4. *Let $(X,\mathcal{F},\mathcal{C})$ and $(X',\mathcal{F}',\mathcal{C}')$ be two Frölicher spaces, with natural diffeologies $\mathcal{P}$ and $\mathcal{P}'$. There is a natural structure of Frölicher space on $X \times X'$ which contours $\mathcal{C} \times \mathcal{C}'$ are the 1-plots of $\mathcal{P} \times \mathcal{P}'$.*

We can even state the same results in the case of infinite products, in a very trivial way by taking the cartesian products of the plots or of the contours. Let us now give the description of what happens for projective limits of Frölicher and diffeological spaces.

Proposition 4.5. *Let Λ be an infinite set of indices, that can be uncoutable.*

- *Let $\{(X_\alpha, \mathcal{P}_\alpha)\}_{\alpha\in\Lambda}$ be a family of diffeological spaces indexed by Λ totally ordered for inclusion, with $(i_{\beta,\alpha})_{(\alpha,\beta)\in\Lambda^2}$ a family of diffeological maps . If $X = \bigcap_{\alpha\in\Lambda} X_\alpha$, X carries the* ***projective diffeology*** *$\mathcal{P}$ which is the pull-back of the diffeologies $\mathcal{P}_\alpha$ of each X_α via the family of maps $(f_\alpha)_{\alpha\in\Lambda}$. The diffeology $\mathcal{P}$ made of plots $g : O \rightarrow X$ such that, for each $\alpha \in \Lambda$,*

$$f_\alpha \circ g \in \mathcal{P}_\alpha.$$

This is the biggest diffeology for which the maps f_α are smooth.

- *Let $\{(X_\alpha, \mathcal{F}_\alpha, \mathcal{C}_\alpha)\}_{\alpha\in\Lambda}$ be a family of Frölicher spaces indexed by Λ totally ordered for inclusion, with $(i_{\beta,\alpha})_{(\alpha,\beta)\in\Lambda^2}$ a family of differentiable maps . with natural diffeologies $\mathcal{P}_\alpha$. There is a natural structure of Frölicher space $X = \bigcap_{\alpha\in\Lambda} X_\alpha$, which contours*

$$\mathcal{C} = \bigcap_{\alpha\in\Lambda} \mathcal{C}_\alpha$$

are the 1-plots of $\mathcal{P} = \bigcap_{\alpha\in\Lambda} \mathcal{P}_\alpha$. A generating set of functions for this Frölicher space is the set of maps of the type:

$$\bigcup_{\alpha\in\Lambda} \mathcal{F}_\alpha \circ f_\alpha.$$

4.1.5. *Differential forms on a diffeological space and differential dimension.*

Definition 4.4. [28] *Let $(X, \mathcal{P})$ be a diffeological space and let V be a vector space equipped with a differentiable structure. A $V-$valued $n-$differential form α on X (noted $\alpha \in \Omega^n(X, V)$) is a map*

$$\alpha : \{p : O_p \rightarrow X\} \in \mathcal{P} \mapsto \alpha_p \in \Omega^n(p; V)$$

such that

- *Let $x \in X$. $\forall p, p' \in \mathcal{P}$ such that $x \in Im(p) \cap Im(p')$, the forms α_p and $\alpha_{p'}$ are of the same order n.*
- *Moreover, let $y \in O_p$ and $y' \in O_{p'}$. If $(X_1, ..., X_n)$ are n germs of paths in $Im(p)\cap Im(p')$, if there exists two systems of $n-$vectors $(Y_1, ..., Y_n) \in (T_yO_p)^n$ and $(Y'_1, ..., Y'_n) \in (T_{y'}O_{p'})^n$, if $p_*(Y_1, ..., Y_n) = p'_*(Y'_1, ..., Y'_n) = (X_1, ..., X_n)$,*

$$\alpha_p(Y_1, ..., Y_n) = \alpha_{p'}(Y'_1, ..., Y'_n).$$

Denote by

$$\Omega(X;V) = \oplus_{n\in\mathbb{N}}\Omega^n(X,V)$$

the set of $V-$valued differential forms.

With such a definition, we feel the need to make two remarks for the reader:

• If there does not exist n linearly independent vectors $(Y_1, ..., Y_n)$ defined as in the last point of the definition, $\alpha_p = 0$ at y.

• Let $(\alpha, p, p') \in \Omega(X,V) \times \mathcal{P}^2$. If there exists $g \in C^\infty(D(p); D(p'))$ (in the usual sense) such that $p' \circ g = p$, then $\alpha_p = g^*\alpha_{p'}$.

Proposition 4.6. *The set $\mathcal{P}(\Omega^n(X,V))$ made of maps $q : x \mapsto \alpha(x)$ from an open subset O_q of a finite dimensional vector space to $\Omega^n(X,V)$ such that for each $p \in \mathcal{P}$,*

$$\{x \mapsto \alpha_p(x)\} \in C^\infty(O_q, \Omega^n(O_p, V)),$$

is a diffeology on $\Omega^n(X,V)$.

Working on plots of the diffeology, one can define the product and the differential of differential forms, which have the same properties as the product and the differential of differential forms.

Definition 4.5. *Let $(X, \mathcal{P})$ be a diffeological space.*

• *$(X, \mathcal{P})$ is **finite-dimensional** at x if and only if*

$$\exists n_0 \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad n \geq n_0 \Rightarrow dim(\Omega^n(X,\mathbb{R})) = 0$$

Then, we set

$$dim(X, \mathcal{P}) = max\{n \in \mathbb{N} | dim(\Omega^n(X,\mathbb{R})) > 0\}.$$

• *If not, $(X, \mathcal{P})$ is called **infinite dimensional**.*

Let us make a few remarks on this definition. If X is a manifold with $dim(X) = n$, the natural diffeology as described in section 4.1.1 (also called “nébuleuse” diffeology) is such that

$$dim(X, \mathcal{P}_0) = n.$$

Now, if $(X, \mathcal{F}, \mathcal{C})$ is the natural Frölicher structure on X, take $\mathcal{P}_1$ generated by the maps of the type $g \circ c$, where $c \in \mathcal{C}$ and g is a smooth map from an open subset of a finite dimensional space to $\mathbb{R}$. This is an easy exercise to show that

$$dim(X, \mathcal{P}_1) = 1.$$

This first point shows that the dimension depends on the diffeology considered. This leads to the following definition, since $\mathcal{P}(\mathcal{F})$ is clearly the diffeology with the biggest dimension associated to $(X, \mathcal{F}, \mathcal{C})$:

Definition 4.6. *The **dimension** of a Frölicher space* $(X, \mathcal{F}, \mathcal{C})$ *is the dimension of the diffeologial space* $(X, \mathcal{P}(\mathcal{F}))$.

4.2. Riemannian diffeological spaces.

Definition 4.7. *Let* $(X, \mathcal{P})$ *be a diffeological space. A **Riemannian metric** g on X (noted* $g \in Met(X)$*) is a map*

$$g : \{p : O_p \to X\} \in \mathcal{P} \mapsto g_p$$

such that

1. $x \in O_p \mapsto g_p(x)$ *is a smooth section of the bundle of symmetric bilinear forms on* TO_p
2. *let* $y \in O_p$ *and* $y' \in O_{p'}$ *such that* $p(y) = p'(y')$*. If* (X_1, X_2) *is a pair of germs of paths in* $Im(p) \cap Im(p')$*, if there exists two systems of* $2-$*vectors* $(Y_1, Y_2) \in (T_y O_p)^2$ *and* $(Y'_1, Y'_2) \in (T_{y'} O_{p'})^2$*, if* $p_*(Y_1, Y_2) = p'_*(Y'_1, Y'_2) = (X_1, X_2)$*,*
$$g_p(Y_1, Y_2) = g_{p'}(Y'_1, Y'_2).$$
3. *for each non zero germ of smooth path* Y*,*
$$g(Y, Y) > 0.$$

$(X, \mathcal{P}, g)$ *is a **Riemannian diffeological space** if g is a metric on* $(X, \mathcal{P})$*. If condition (3) is not everywhere fulfilled, we call it **pseudo-Riemannian diffeological space.***

For any germ of path X we note $||X|| = \sqrt{g(X, X)}$.

Definition 4.8. *We call **arc length** the map* $L : C^\infty([0;1], X) \to \mathbb{R}_+$ *defined by*

$$L(\gamma) = \int_0^1 ||\dot{\gamma}(t)|| dt.$$

Let $(x, y) \in X^2$*. We define*

$$d_g(x, y) = \inf \{L(\gamma) | \gamma(0) = x \wedge \gamma(1) = y\}$$

*and we call **Riemannian pseudo-distance** the map* $d : X \times X \to \mathbf{R}_+$ *that we have just described.*

The following proposition justifies the terms "pseudo-distance":

Proposition 4.7. (1) $\forall x \in X, d_g(x,x) = 0$.
(2) $\forall (x,y) \in X^2, d_g(x,y) = d_g(y,x)$
(3) $\forall (x,y,z) \in X^3, d_g(x,z) \leq d_g(x,y) + d_g(y,z)$.

Proof. The proofs are standard, let us recall the main arguments. For 1, the constant path gives the minimum. For 2, the reverse parametrization $t \mapsto 1-t$ defines a transformation from the paths from x to y to the paths from y to x under which L is invariant. For 3, the paths passing by y are only a part of the set of paths from x to z.

One could wonder whether d is a distance or not, i.e. if we have the stronger property:

$$\forall (x,y) \in X^2, d_g(x,y) = 0 \Leftrightarrow x = y.$$

Unfortunately, it seems to appear in examples arising from infinite dimensional geometry that there can have a distance which equals to 0 for $x \neq y$. This is what is described on e.g. a weak Riemannian metric of a space of proper immersions in the work of Michor and Mumford [24]. Moreover, the D-topology is not the topology defined by the pseudo-metric d. All these facts, which show that the situation on Riemannian diffeological spaces is different from the one known on finite dimensional manifolds, are checked in the following remark.

Remark 4.3. *Let $Y = \coprod_{i \in \mathbf{N}^*} \mathbf{R}_i$, where $\mathbf{R}_i$ is the $i-$th copy of $\mathbf{R}$, equipped with its natural scalar product. Let $\mathcal{R}$ be the equivalence relation*

$$x_i \mathcal{R} x_j \Leftrightarrow \begin{cases} (x_i \notin]0;\frac{1}{i}[\wedge x_j \notin]0;\frac{1}{j}[) \Rightarrow \begin{cases} x_i = x_j & \text{if } x_i \leq 0 \\ x_i + 1 - \frac{1}{i} = y_j + 1 - \frac{1}{j} & \text{if } x_i \geq \frac{1}{i} \end{cases} \\ (x_i \in]0;\frac{1}{i}[\vee x_j \in]0;\frac{1}{j}[) \Rightarrow i = j \wedge x_i = x_j \end{cases}$$

Let $X = Y/\mathcal{R}$. This is a 1-dimensional Riemannian diffeological space. Let $\bar{0}$ be the class of $0 \in \mathbf{R}_1$, and let $\bar{1}$ be the class of $1 \in \mathbf{R}_1$. Then $d_g(\bar{0},\bar{1}) = 0$. This shows that d_g is not a distance on X. In the $D-$topology, $\bar{0}$ and $\bar{1}$ respectively have the following disconnected neighborhoods:

$$U_{\bar{0}} = \left\{ \bar{x}_i | x_i < \frac{1}{2i} \right\}$$

and

$$U_{\bar{1}} = \left\{ \bar{x}_i | x_i > \frac{1}{2i} \right\}.$$

This shows that d does not define the $D-$topology.

This leads to the following definition:

Definition 4.9. *A* ***Frölicher-strong*** *or* ***arc-length-strong*** *Riemannian metric is a Riemannian metric g for which d is a distance.*

We have to notice that this notion is not the same as the notion of strong Riemannian metric on a Hilbert vector bundle. Based on the arc-length pseudo-distance d; the terminology "arc-length strong" appears to us as very natural. However, since (smooth) paths are the contours of the Frölicher structure, and since they generate the Frölicher structure itself, we get the intuition that the notion of arc-length strong metric is itself intrinsically related to the Frölicher structure under consideration. This is the reason why we propose also the terminology "Frölicher-strong" that we shall use all along the text.

4.3. **Volume and diffeologies.** On a (finite dimensional) Riemannian manifold M, the notion of Riemannian volume is related to the dimension of the manifold, and to the notion of volume form ω_M. on the one hand, if the Riemannian manifold M is viewed now as a Frölicher space, with underlying diffeology $\mathcal{P}$, we have that

$$\forall f \in \mathcal{P}, f^*\omega_M = 0 \Leftrightarrow \text{ Dimension of } f < dim M.$$

On the other hand, if $f \in \mathcal{P}$ is an embedding $O \rightarrow M$, and if O is an open domain of dimension p, $f(O)$ is a submanifold of M and it can be equipped with the $p-$dimensional Hausdorff measure induced by the geodesic distance on O, and we have:

Proposition 4.8. *Assume that $O \subset \mathbf{R}^m$ is a $n-$dimensional submanifold. The Hausdorff dimension of O is n and, for any relatively open subset $U \subset O$, if $\mathcal{H}^n$ is the $n-$dimensional Hausdorff measure in $\mathbf{R}^m$,*

$$\mathcal{H}^n(U) = \int_U \omega_O.$$

We remark that, given a chart O on M, O is equipped with the standard Lebesgue volume $d\lambda = \omega_O$, and M is equipped with the Riemannian volume ω_M, on O,

$$\omega_M = \sqrt{detg}.d\lambda$$

and hence, if U is a n-dimensional submanifold of O, noting i_u the canonical injection $U \rightarrow O$, we can define

$$\mathcal{H}^n_M(U) = \int_U \sqrt{det(i^*g)} d\mathcal{H}^n.$$

This corresponds to the n dimensional Hausdorff measure with respect to M viewed as a metric space. As a consequence, a Riemannian manifold does not only carry one volume

measure, but a family of measures on the plots of its diffeology. If f is a $n$dimensional plot of the diffeology of M, we define $\mathcal{H}_f = f^*\mathcal{H}^n_M$. This property is stable under composition of plots, reparametrization, gluing. This leads to the following on Riemannian diffeological spaces:

Definition 4.10. *Let $(X,\mathcal{P})$ be a diffeological space. The **Hausdorff diffeological volume** associated to a Riemannian metric is the collection $\{\mathcal{H}_p; p \in \mathcal{P}\}$ of $dim(D(p))$ Hausdorff measures on the domains $D(p)$.*

Let us remark that:

- if $det(p^*g) > 0$ on $D(p)$, the domain of p, then $\mathcal{H}_p$ is $dim(D(p))$-dimensional Hausdorff measures on $D(p)$ induced by the Riemannian distance on $D(p) \subset X$.
- for any $(p,p') \in \mathcal{P}^2$ of same dimension, if $p' = f \circ p$, $\mathcal{H}'_p = f^*\mathcal{H}_p$.
- If there exists $x \in D(p)$ such that $det(p^*g_x) = 0$, the definition of the Hausdorff metric via the (pseudo)-distance on $D(p)$ remains valid [15].

We have here to remark that the Riemannian metric needs not to be Frölicher-strong, because this is the induced Riemannian metric on each $D(p)$ which defines the Hausdorff measure, one should say on the (classical) Riemannian manifold $D(p)$.

4.4. **On $\infty - p$ forms and volume forms.** This section is based on ideas from A. Asada [1, 2] adapted to the context of a diffeological space .

Definition 4.11. *Let $(X,\mathcal{P})$ be a Riemannian diffeological space. An **orientable plot** on X is a plot $p \in \mathcal{P}$, $dim(p) = n$, such that there exists a n-form $\omega_n \in \Omega^n(X,\mathbf{R})$ such that $p_*h_p = \omega_n$, where h_p is a $n-$form on $D(p)$, that induces the Hausdorff measure $\mathcal{H}_p$. The space of orientable plots of the diffeology $\mathcal{P}$ is noted $\mathcal{O}(\mathcal{P})$.*

Proposition 4.9. *Let X be a smooth n-dimensional compact manifold, equipped with its nébuleuse diffeology $\mathcal{P}$. X is orientable if and only if there exists a n-plot $p \in \mathcal{P}$, surjective, such that $p \in \mathcal{O}(\mathcal{P})$.*

The proof is straightforward, setting a Riemannian metric g on X, and using the exponential map to define the plot p.

Example 4.1. *Let $X = S^n$, $n \geq 1$. Let P, P' be two antipodal points. The mapping $exp_P : T_PS^n \to S^n$ has an injectivity radius $r = \pi$. The cut locus is P'. Thus, considering the*

open ball $B(0, 3\pi/2) \subset T_P S^n$ and the plot $p : exp_P|_{B(0,3\pi/2)}$, we get the construction, even if $p^*\omega_{S^n} = 0$ on $p^{-1}(P')$ (here, ω_{S^n} is the canonical volume form on S^n).

Example 4.2. *Let $X' =]0;1[\times[0;1]$, let $\sim$ be the relation of equivalence on X' defined by $(t, 0) \sim (1-t, 1)$ and let $X = X'/\sim$ be the (open) Möbius band. The mapping $p :]0;1[\times]-1/2;3/2[\to X'$ defined by*

$$p(x,y) = \begin{cases} (x, y+1) & if \quad x < 0 \\ (x,y) & if \quad x \in [0;1] \\ (x, y-1) & if \quad x > 1 \end{cases}$$

is a 2-dimensional plot such that the trace of the canonical Lebesgue measure coïncides with $\mathcal{H}_p$, but for which there exists no 2-form ω_2 on X' such that $p_*\lambda = \omega_2$ where λ is the canonical Lebesgue measure on $]0;1[\times]-1/2;3/2[$, because ω_2 should be non zero everywhere.

After these examples, let us turn to the key definition :

Definition 4.12. *Let $(X, \mathcal{P})$ be a Riemannian diffeological space with set of oriented plots $\mathcal{OP}$. We call* ***volume form*** *of X a collection of forms*

$$p \in \mathcal{OP} \mapsto \omega_p \in \Omega^{dimD(P)}(D(p), \mathbf{R})$$

such that

- the form ω_p defines the $dim(D(p))-$ Hausdorff measure on $D(p)$

- if p and p' are oriented $n-$plots such that $p' = p \circ f$, then $\omega'_p = f^*\omega_p$

Definition 4.13. *Let $(X, \mathcal{P})$ be a Riemannian diffeological space with volume form ω. Let $q \in \mathbf{N}$. A $(\infty - q)-$form is a collection*

$$p \in \mathcal{OP} \mapsto \omega_p \in \Omega^{dimD(P)-q}(D(p), \mathbf{R})$$

such that there exists a $q-$form $\beta \in \Omega^q(X, \mathbf{R})$ such that

$$\forall p \in \mathcal{OP}, \alpha_p \wedge p^*\beta = \omega_p.$$

For the consistency of the definition, if $q > dim(p)$ or if $\omega_p = 0$, we set $\alpha_p = 0$.

With such a definition, a volume form is a $(\infty - 0)-$form. We note by $\Omega^{(\infty-q)}(X, \mathbf{R})$ the space of $(\infty - q)-$forms.

5. $H^s(S^1,N)$ as a Riemannian diffeological space

5.1. Settings.

Proposition 5.1. *Let $s \leq 1/2$.. Then $H^s(S^1,N)$ and $H^s_0(S^1,N)$ are Riemannian Frölicher spaces. The same holds for $N = G$.*

Proof. $H^s(S^1,\mathcal{M})$ is equipped with its natural underlying structure of Hilbert space, which carries a natural structure of Frölicher space. As subsets, $H^s(S^1,N)$ and $H^s_0(S^1,N)$ are equipped with the reflexive completion of their trace diffeology. So that, they are Frölicher spaces. The natural Hilbert structure on $H^s(S^1,\mathcal{M})$ induces a Riemannian metric on $H^s(S^1,N)$.

Proposition 5.2. *$H^s(S^1,N)$ is Frölicher-strong for $s \in \mathbf{R}$.*

Proof. Let γ be a smooth path in $H^s(S^1,N) \subset H^s(S^1,\mathcal{M})$. Then the length of γ is bounded below by $||\gamma(1) - \gamma(0)||_{H^s(S^1,\mathcal{M})}$

Let us now give a result for the extension of the multiplication of loop groups. For this, we define the space

$$H^{1/2,+}(S^1,G) = \bigcup_{s>1/2} H^s(S^1,G)$$

Lemma 5.1. *The space $H^{1/2,+}(S^1,G)$ is a Lie group modeled on a locally convex vector space.*

Proof. For each $s > 1/2$, we have $H^s(S^1,G) \subset C^0(S^1,G)$ and the usual (exponential) atlas on $H^s(S^1,G)$ is induced by the atlas on $C^0(S^1,G)$, see [14, 27] for the details. Then, following [18], we get the result.

Proposition 5.3. *Let $k > 1/2$ and let $s \leq k$. The natural action $C^\infty(S^1,G) \times C^\infty(S^1,G) \to C^\infty(S^1,G)$ extends to a smooth action*

$$H^k(S^1,G) \times H^s(S^1,G) \to H^s(S^1,G),$$

and for $s \leq 1/2$, to a smooth action

$$H^{1/2,+}(S^1,G) \times H^s(S^1,G) \to H^s(S^1,G).$$

Proof. Since $G \subset \mathcal{M}$, with smooth inclusion and trace diffeology, it is sufficient to remark that this theorem is an application of the standard "multiplication theorem' of

Sobolev classes, which states that the multiplication, with the coefficients defined as above, is a bilinear continuous map.

5.2. $H_0^{1/2}(S^1, N)$ **and symplectic forms.** We use here the following property: let $\int_{S^1}(.;.)$ be the L^2 scalar product. Then this scalar product coincides with the duality pairing between H_0^s and $H^{-s} = (H_0^s)'$ (topological dual) for $s > 0$ on $C^\infty(S^1, \mathcal{M})$.

Let $\gamma \in H_0^{1/2}(S^1, \mathcal{M})$. Then:

Lemma 5.2. *$\dot\gamma \in H^{-1/2}(S^1, \mathcal{M})$ and the canonical 1-form*

$$\theta(X) = \int_{S^1} (\dot\gamma, X)$$

defined first for $\gamma \in C_0^\infty(S^1, \mathcal{M})$ and $X \in C^\infty(S^1, \mathcal{M})$ extends to a 1-forms on $H_0^{1/2}(S^1, \mathcal{M})$.

Proof. Since $\gamma \in H_0^{1/2}(S^1, \mathcal{M})$, differentiation is a differential operator of order 1 and hence $\dot\gamma \in H^{1/2-1}(S^1, \mathcal{M}) = H^{-1/2}(S^1, \mathcal{M})$ and the map $\gamma \mapsto \dot\gamma$ is smooth. Let us recall that $H_0^{1/2}(S^1, \mathcal{M})$ is a vector space, and in particular a flat Hilbert manifold. Its tangent bundle is then identified canonically with the product:

$$TH_0^{1/2}(S^1, \mathcal{M}) = H_0^{1/2}(S^1, \mathcal{M}) \times H_0^{1/2}(S^1, \mathcal{M}).$$

By the way, a tangent vector X at $\gamma \in H_0^{1/2}(S^1, \mathcal{M})$ is identified with an element of $H_0^{1/2}(S^1, \mathcal{M})$. By the pairing of dual spaces

$$H^{-1/2} \times H_0^{1/2} \to \mathbf{R},$$

the formula

$$\theta(X) = \int_{S^1} (\dot\gamma, X)$$

extends smoothly for $(\gamma, X) \in \left(H_0^{1/2}(S^1, \mathcal{M})\right)^2$.

So that,

Theorem 5.1. *The symplectic 2-form on the based loop space $C_0^\infty(S^1, N)$ defined by*

$$\omega_N(X, Y) = d\theta(X, Y) = \int_{S^1} \left(\frac{\nabla^N}{ds} X(s), Y(s)\right) ds$$

extends to a 2-form on $H_0^{1/2}(S^1, N)$.

Proof. The 1-form $\theta \in \Omega^1(H_0^{1/2}(S^1, \mathcal{M}), \mathbf{R})$ restricts to a 1-form on $H_0^{1/2}(S^1, N)$. Following [28, 19], the 2-form $\omega_N = d\theta$ is well-defined on $H_0^{1/2}(S^1, N)$ and restricting to $C_0^\infty(S^1, N)$, by the formula $\omega_N = d\theta$ which still holds, we recover the usual symplectic form of the based loop space [30].

This is also the case when $N = G$. On $C_0^\infty(S^1,G)$, the vector field $\dot\gamma$ is not left-invariant. We know that, on the based loop group, there is another symplectic form, which is not exact, defined for left-invariant vector fields X and Y by

$$\omega_G(X,Y) = \int_{S^1} \left(\frac{dX(s)}{ds}, Y(s) \right) ds.$$

But the 2-form ω_G does not seem to extend to $H_0^{1/2}(S^1,G)$ because the full space $H_0^{1/2}(S^1,G)$ is not a group. The biggest known group in $H_0^{1/2}(S^1,G)$ is $H_0^{1/2}(S^1,G) \cap C^0(S^1,G)$, see [27].

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EXISTENCE FOR STOCHASTIC COUPLED SYSTEMS ON NETWORKS WITH TIME-VARYING DELAY DRIVEN BY ROSENBLATT PROCESS WITH DELAY AND POISSON JUMPS

TAYEB BLOUHI* AND MOHAMED FERHAT

ABSTRACT. We present some results on the existence and uniqueness of mild solutions for system of semilinear impulsive differential with infinite fractional Brownian motions. Our approach is based on Perov's fixed point theorem and a new version of Schaefer's fixed point theorem in generalized Banach spaces. Also, we investigate the relationship between mild and weak solutions.

1. INTRODUCTION

Differential equations with impulses were considered for the first time by Milman and Myshkis [18] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [13]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses.

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to the basic theory is well developed in the monographs by Benchohra et al [4], Graef *et al* [9], Laskshmikantham *et al.* [2], Samoilenko and Perestyuk [25].

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems; see for instance the monographs by Da Prato and Zabczyk [24], Gard [10], Gikhman and Skorokhod [11], Sobzyk [29] and Tsokos and Padgett [30]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [30] to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs by Bharucha-Reid [3], Mao[17], Øksendal, [21], Tsokos and Padgett [30], Sobczyk [29] and Da Prato and Zabczyk [24].

The study of impulsive stochastic differential equations is a new research area. The existence and stability of stochastic of impulsive of differential equations were recently investigated, for example in [8, 9, 14, 15, 16, 23, 22, 31, 32].

This paper is concerned with a system of the following neutral stochastic partial differential equations with delay driven by a Rosenblatt process of the form:

$$
\begin{cases}
d(x(t) & +g^1(t,x(t-u(t)),y(t-u(t))) = (A_1x(t) \\
& +f^1(t,x(t-r(t)),y(t-r(t)))dt+\sigma^1(t))dZ_1^H(t) \\
& +\displaystyle\int_{\mathcal{Z}} h^1(t,x(t-\rho(t)),y(t-\rho(t)),\kappa)\widetilde{N}(dt,d\kappa),\ t\in[0,b], t\neq t_k, \\
d(y(t) & +g^2(t,x(t-u(t)),y(t-u(t))) = (A_2x(t) \\
& +f^2(t,x(t-r(t)),y(t-r(t)))dt+\sigma^2(t))dZ_2^H(t) \\
& +\displaystyle\int_{\mathcal{Z}} h^2(t,x(t-\rho(t)),y(t-\rho(t)),\kappa)\widetilde{N}(dt,d\kappa),\ t\in[0,b], t\neq t_k, \\
\Delta x(t) & = I_k^1(x(t_k),y(t_k)), \quad k=1,2,\dots,m \\
\Delta y(t) & = \overline{I}_k^2(y(t_k),y(t_k)), \\
x(t) & = \phi_1(t),\ -\tau\leq t\leq 0 \\
y(t) & = \phi_2(t),\ -\tau\leq t\leq 0
\end{cases}
\tag{1.1}
$$

Here, $x(\cdot), y(\cdot)$ takes the value in the separable Hilbert space X with inner product $\langle\cdot,\cdot\rangle$ induced by the norm $\|\cdot\|$, $A_i : D(A_i)\subset X \longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S_i(t))_{t\geq 0}$ in X for each $i = 1,2$ and

$f^i, g^i : [0,b] \times X \times X \longrightarrow X$, $Z^H(t)$ is a Rosenblatt process on a real and separable Hilbert space Y with parameter $H \in (\frac{1}{2}, 1)$, $u(t), r(t) : J \to [0,\tau](\tau > 0)$ are continuous, $\sigma_l^1, \sigma_l^2 : J \to L_Q^0(Y,X)$. Here, $L_Q^0(Y,X)$ denotes the space of all Q_i-Hilbert-Schmidt operators from Y into X, which will be defined in the next section. $I_k, \bar{I}_k \in C(X \times X, X)$ $(k = 1,2,\dots,m)$, $h^1, h^2 : J \times X \times X \times \mathcal{U} \to X$, which will be also defined in the next section (see section 2 below). Moreover, the fixed times t_k satisfies $0 < t_1 < t_2 < \dots < t_m < b$, $y(t_k^-)$ and $y(t_k^+)$ denotes the left and right limits of $y(t)$ at $t = t_k$. As for x we mean the segment solution which is defined in the usual way, that is, if $x(\cdot,\cdot) : [-\tau, b] \times \Omega \to X$, then for any $t \geq 0$. Let $\mathcal{D}_{\mathcal{F}_0}$ be the following space defined by

$$\mathcal{D}_{\mathcal{F}_0} = \big\{\phi_i \colon [-\tau, 0] \times \Omega \to X \text{ is continuous everywhere except for a finite number of points } \phi(t_k^-) \text{ and } \phi(t_k^+) \text{ with } \phi(t_k) = \phi(t_k^-)\big\},$$

endowed with the norm

$$||\phi(t)||_{\mathcal{D}_{\mathcal{F}_0}} = \int_{-\tau}^0 |\phi(t)|^2 dt.$$

Now, for a given $b > 0$, we define

$$\mathcal{D}_{\mathcal{F}_b} = \big\{x \colon [-\tau, b] \times \Omega \to X,\ x_k \in C(J_k, X) \text{ for } k = 1, \dots m,\ \phi_i \in \mathcal{D}_{\mathcal{F}_0}, \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+) \text{ with } x(t_k) = x(t_k^-),\ \ k = 1, \cdots, m, \text{ and } \mathbb{E}(\sup_{t \in [0,b]} \|y(t)\|^2) < \infty\big\},$$

endowed with the norm

$$\|x\|_{\mathcal{D}_{\mathcal{F}_b}} = \mathbb{E}(\sup_{0 \leq s \leq T} \|x(s)\|^2)^{\frac{1}{2}},$$

where x_k denotes the restriction of x to $J_k = (t_{k-1}, t_k]$, $k = 1, 2, \cdots, m$, and $J_0 = [-\tau, 0]$.

$$\begin{cases} dz(t) + g_*(t, z(t-u(t)) = A_* z(t) + f(t, z(t-r(t)))dt + \sigma^1(t) dZ^H(t) t_k \\ \qquad + \displaystyle\int_{\mathcal{Z}} h(t, z(t-\rho(t)), \kappa) \widetilde{N}(dt, d\kappa),\ t \in [0,b], t \neq t_k, \\ \Delta z(t) = I_k^*(z(t_k)), \quad t = t_k \quad k = 1, 2, \dots, m \\ z(t) = \phi(t),\ -\tau \leq t \leq 0 \end{cases} \tag{1.2}$$

where

$$z(t-u(t)) = \begin{bmatrix} x(t-u(t)) \\ y(t-u(t)) \end{bmatrix}, A_* = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},\ f(t, z(t-r(t)) = \begin{bmatrix} f^1(t, x(t-r(t)), y(t-r(t))) \\ f^2(t, x(t-r(t)), y(t-r(t))) \end{bmatrix}$$

and

$$\sigma(t) = \begin{bmatrix} \sigma^1(t) \\ \sigma^2(t) \end{bmatrix},\ g(t, z(t-r(t)) = \begin{bmatrix} g^1(t, x(t-u(t)), y(t-u(t))) \\ g^2(t, x(t-u(t)), y(t-u(t))) \end{bmatrix}\ \phi(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}.$$

and

$$h(t,z(t-\rho(t)),\kappa) = \begin{bmatrix} h^1(t,x(t-\rho(t),y(t-\rho(t)),\kappa) \\ h^2(t,x(t-\rho(t),y(t-\rho(t)),\kappa) \end{bmatrix}, I_k^*(z(t_k)) = \begin{bmatrix} I_k^1(x(t_k),y(t_k)) \\ I_k^2(x(t_k),y(t_k)) \end{bmatrix}$$

Some results on the existence of solutions for differential equations with infinite Brownian motion were obtained in [12, 31]. Some existence and uniqueness of mild solutions to neutral stochastic delay functional integro-differential equations perturbed by a fractional Brownian motion can be found in Caraballo and Diop [7].

This paper is organized as follows. In Section 2, we summarize several important working tools on Rosenblatt process, Poisson point processes and we recall some preliminary results about analytic semi-groups and fractional power associated to its generator that will be used to develop our results. In section 3, by Perov's fixed point theorem we consider a sufficient condition for the existence, uniqueness and exponential decay to zero in mean square for mild solutions of equation (1.1). In Section 4, we give an example to illustrate the efficiency of the obtained result.

2. Preliminaries

In this section, we introduce some notations, and recall some definitions, and preliminary facts which are used throughout this paper. Actually we will borrow it from [19, 5]. Although we could simply refer to this paper whenever we need it, we prefer to include this summary in order to make our paper as much self-contained as possible.

Let $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all P-null sets. Suppose $\{p(t), t\geq 0\}$ is a σ-finite stationary $\mathcal{F}_t$-adapted Poisson point process taking values in a measurable space $(\mathcal{U},\mathcal{B}(\mathcal{U}))$. The random measure N_p defined by $N_p((0,t]\times\Lambda) := \sum_{s\in(0,t]} 1_\Lambda(p(s))$, for $\Lambda\in\mathcal{B}(\mathcal{U})$ is called the Poisson random measure induced by $p(.)$, thus, we can define the measure $\widetilde{N}.$ by $\widetilde{N}(dt,d\kappa) := N_p(dt,d\kappa) - \nu(dz)dt$, where ν is the characteristic measure of N_p, which is called the compensated Poisson random measure, for a Borel set $\mathcal{Z}\in\mathcal{B}(\mathcal{U}-\{0\})$.

2.1. **Rosenblatt process.** We briefly recall the Rosenblatt process as well as the Wiener integral with respect to it.

Consider $(\xi_n)_{n\in\mathbf{Z}}$ a stationary Gaussian sequence with mean zero and variance 1 such that its correlation function satisfies that $R(n) := E(\xi_0\xi_n) = n^{\frac{2H-2}{k}}L(n)$, with $H\in(\frac{1}{2},1)$ and L

is a slowly varying function at infinity. Let G be a function of Hermite rank k, that is, if G admits the following expansion in Hermite polynomials

$$G(x)=\sum_{j\geq 0}c_jH_j(x), \qquad c_j=\frac{1}{j!}E(GE(\xi_0)H_j(\xi_0)),$$

and

$$H_j(x)=(-1)^je^{\frac{x^2}{2}}\frac{d^j}{dx^j}e^{\frac{-x^2}{2}}$$

where $H_j(x)$ is the Hermite polynomial of degree j, then $k=\min\{\mid j\mid, c_j\neq 0\}\geq 1$, the Non-Central Limit Theorem , $\frac{1}{n^H}\sum_{j=1}^{[nt]}G(\xi_J)$ converges as $n\to\infty$, in the sense of finite dimensional distributions, to the process

$$Z_k^H=c(H,k)\int_{\mathbf{R}^k}\int_0^t\Big(\prod_{j=1}^{k}(s-y_j)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\Big)dsdB(\theta_{1)}\dots B(\theta_{k)} \tag{2.3}$$

where the above integral is a Wiener.Ito multiple integral of order k with respect to the standard Brownian motion $(B(\theta))_{\theta\in\mathbf{R}}$ and $c(H,k)$ is a positive normalization constant depending only on H and k. The process $(Z_k^H(t))_{t\geq 0}$ is called as the Hermite process and it is H self-similar in the sense that for any $c>0$, $(Z_k^H(ct)=c^HZ_k^H(t))$ and it has stationary increments [1].

When $k=1$ the Hermite process given by (2.3) is the fractional Brownian motion with Hurst parameter $H\in(\frac{1}{2},1)$ [34]. If $k=2$ then the process (2.3) is called as the Rosenblatt process which arises from the Non-Central Limit Theorem (see [35] and references therein). Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{Z^H(t)\ t\in[0,T]\}$ be a one-dimensional Rosenblatt process with parameter $H\in(\frac{1}{2},1)$. By Tudor [36], the Rosenblatt process with parameter $H>\frac{1}{2}$ can be written as

$$Z^H(t)=d(H)\int_0^t\int_0^t\int_0^t\Big[\int_{\theta_1\vee\theta_2}^t\frac{\partial K^{H'}}{\partial u}(u,\theta_1)\frac{\partial K^{H'}}{\partial u}(u,\theta_2)du\Big]dB(\theta_1)dB(\theta_2), \tag{2.4}$$

where $K^H(t,s)$ is given by

$$K_H(t,s)=c_H\int_s^t(u-s)^{H-\frac{3}{2}}\left(\frac{u}{s}\right)^{H-\frac{1}{2}}du, \quad t\geq s,$$

where $c_H=\sqrt{\frac{H(2H-1)}{\Gamma(2H-2,H-\frac{1}{2})}}$ and $\Gamma(\cdot,\cdot)$ denotes the Beta function. We put $K^H(t,s)=0$ if $t\leq s$.

$$\frac{\partial K_H}{\partial t}(t,s)=c_H\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}.$$

where $(B(t), t \in [0,T])$ is a Brownian motion, $H^{'} = \frac{H+1}{2}$ and $d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}}$ is a normalizing constant. The covariance of the Rosenblatt process $\{Z^H(t), t \in [0,T]\}$ satisfies that $R_H(t,s) = E[Z^H(t))Z^H(s)]$

$$R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \quad t, s \in [0,T],$$

One note that

$$Z^H(t) = \int_0^T \int_0^T I(\chi_{[0,t)})(\theta_1, \theta_2) dB(\theta_1) dB(\theta_2)$$

where the operator I is defined on the set of functions $f : [0,T] \to \mathbf{R}$, which takes its values in the set of functions $G : [0,T]^2 \to \mathbf{R}^2$ and is given by

$$I(f)(\theta_1, \theta_2) = d(H) \int_{\theta_1 \vee \theta_2}^T f(u) \frac{\partial K^{H^{'}}}{\partial u}(u, \theta_1) \frac{\partial K^{H^{'}}}{\partial u}(u, \theta_2) du$$

Let f be an element of the set $\mathcal{E}$ of step functions on $[0,T]$ of the form

$$f = \sum_{i=1}^{n-1} a_i \chi_{(t_i, t_{i+1}]}, \;\; t_i \in [0,T]$$

Then, it is natural to define its Wiener integral with respect to Z^Has

$$\int_0^T f(u) Z^H(u) := \sum_{i=1}^{n-1} a_i (Z^H(t_{i+1}) - Z^H(t_i)) = \int_0^T \int_0^T I(f)(\theta_1, \theta_2) dB(\theta_1) dB(\theta_2)$$

Let $\mathcal{H}$ be the set of functions f such that

$$\mathcal{H} = \left\{ f : [0,T] \to \mathbf{R} : \quad \|f\|^2_{\mathcal{H}} = 2 \int_0^T \int_0^T (I(f)(\theta_1, \theta_2))^2 d(\theta_1) d(\theta_2) < \infty \right\}$$

It follows that (see [36])

$$\|f\|^2_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T f(u) f(v) |u-v|^{2H-2} du dv.$$

It is shown in [1] that the mapping

$$f \to \int_0^T f(u) dZ^H(u)$$

defines an isometry from $\mathcal{E}$ to $L^2(\Omega)$ and it can be extended continuously to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ because $\mathcal{E}$ is dense in $\mathcal{H}$. We call this extension as the Wiener integral of $f \in \mathcal{H}$ with respect to Z^H.

We refer to [36] for the proof of the fact that K.H is an isometry between H and $L^2([0,T])$.

It follows from [36] that H contains not only functions but its elements could be also distributions. In order to obtain a space of functions contained in H, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions f such that

$$\|f\|^2_{|\mathcal{H}|} = H(2H-1)\int_0^T\int_0^T |f(u)||f(v)||u-v|^{2H-2}dudv$$

where $\alpha_H = H(2H-1)$. The space $|H|$ is a Banach space with the norm $\|f\|_{|\mathcal{H}|}$ and we have the following inclusions (see[36]).

As a consequence, we have

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}$$

For any $f \in L^2([0,T])$, we have

$$\|f\|^2_{|\mathcal{H}|} = 2HT^{2H-1}\int_0^T |f(s)|^2 ds$$

and

$$\|f\|^2_{|\mathcal{H}|} \leq C(H)\|f\|^2_{L^{\frac{1}{H}}([0,T])}$$

for some constant $C(H) > 0$. For simplicity throughout this paper we let $C(H) > 0$ stand for a positive constant depending only on H and its value may be different in different appearances.

Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$(K_H^* f)(\theta_1,\theta_2) = \int_{\theta_1\vee\theta_2}^T f(t)\frac{\partial\mathcal{K}}{\partial t}(t,\theta_1,\theta_2)dt,$$

where $\mathcal{K}$ is the kernel of Rosenblatt process in representation (2.4)

$$\mathcal{K}(t,\theta_1,\theta_2) = \chi_{[0,t]}(\theta_1)\chi_{[0,t]}(\theta_2)\int_{\theta_1\vee\theta_2}^T \frac{\partial K^{H'}}{\partial u}(u,\theta_1)\frac{\partial K^{H'}}{\partial u}(u,\theta_2)du$$

Notice that $(K_H^*\chi_{[0,t]})(\theta_1,\theta_2) = \mathcal{K}(t,\theta_1,\theta_2)\chi_{[0,t]}(\theta_1)\chi_{[0,t]}(\theta_2)$. The operator K_H^* is an isometry between $\mathcal{E}$ to $L^2([0,T])$, which could be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have

$$\begin{aligned}
\langle K_H^*\chi_{[0,t]}, K_H^*\chi_{[0,s]}\rangle_{L^2([0,T])} &= \langle \mathcal{K}(t,.,.)\chi_{[0,t]}, \mathcal{K}(t,.,.)\chi_{[0,s]}\rangle_{L^2([0,T])} \\
&= \int_0^{t\wedge s}\int_0^{t\wedge s} \mathcal{K}(t,\theta_1,\theta_2)\mathcal{K}(s,\theta_1,\theta_2)d\theta_1 d\theta_2 \\
&= H(2H-1)\int_0^t\int_0^s |u-v|^{2H-2}dudv \\
&= \langle \chi_{[0,t]}, \chi_{[0,s]}\rangle_{\mathcal{H}}.
\end{aligned}$$

Moreover, for $f \in \mathcal{H}$, we have

$$Z^H(f) = \int_0^T \int_0^T K_H^*(f)(\theta_1, \theta_2) dB(\theta_1) dB(\theta_2)$$

Let $\{z_n(t)\}_{n\in\mathbf{N}}$ be a sequence of two-sided one dimensional Rosenblatt process mutually independent on $(\Omega, \mathcal{F}, P)$. We consider a K-valued stochastic process $Z_Q(t)$ given by the following series:

$$Z_Q(t) = \sum_{n=1}^{\infty} z_n(t) Q^{1/2} e_n, \quad t \geq 0$$

Moreover, if Q is a non-negative self-adjoint trace class operator, then this series converges in the space K, that is, it holds that $Z_Q(t) \in L^2(\Omega, K)$. Then, we say that the above $Z_Q(t)$ is a K-valued $Q-$ Rosenblatt process with covariance operator Q. For example, if $\{\sigma_n\}_{n\in\mathbf{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, assuming that Q is a nuclear operator in K, then the stochastic process

$$Z_Q(t) = \sum_{n=1}^{\infty} z_n(t) Q^{\frac{1}{2}} e_n = \sum_{n=1}^{\infty} z_n(t) \sqrt{\sigma_n} e_n, \;\; t \geq 0$$

is well-defined as a X-valued $Q-$ Rosenblatt process.

Definition 2.1. *Let $\phi : [0,T] \to L_Q^0(Y,X)$ such that $\sum_{n=1}^{\infty} \|K_H^*(\phi Q^{\frac{1}{2}} e_n)\|_{L^2([0,T],X)} < \infty$. Then, its stochastic integral with respect to the Rosenblatt process $Z_Q(t)$ is defined, for $t \geq 0$, as follows:*

$$\int_0^t \phi(s) dZ_Q(s) := \sum_{n=1}^{\infty} \int_0^t \phi(s) Q^{1/2} e_n dz_n(s) = \int_0^t K_H^*(\phi Q^{\frac{1}{2}} e_n)(\theta_1, \theta_2) dB(\theta_1) dB(\theta_2) \quad (2.5)$$

Now, we end this subsection by stating the following result which is fundamental to prove our result.

Lemma 2.1. [6] *For any $\phi : [0,T] \to L_Q^0(Y,X)$ such that $\sum_{n=1}^{\infty} \|\phi Q^{\frac{1}{2}} e_n\|_{L^{\frac{1}{H}}([0,T],X)}$ holds, and for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$E \left\| \int_\alpha^\beta \phi(s) dZ_Q(s) \right\|^2 \leq c_H H(2H-1)(\alpha-\beta)^{2H-1} \sum_{n=1}^{\infty} \int_\alpha^\beta \left\| \phi(s) Q^{1/2} e_n \right\|^2 ds. \quad (2.6)$$

where $c = c(H)$. If, in addition,

$$\sum_{n=1}^{\infty} \|\phi Q^{1/2} e_n\| \text{ is uniformly convergent for } t \in [0,T]$$

then

$$E \left\| \int_\alpha^\beta \phi(s) dB_l^H(s) \right\|^2 \leq c_H H(2H-1)(\alpha-\beta)^{2H-1} \int_\alpha^\beta \|\phi(s)\|_{L_Q^0}^2 ds. \quad (2.7)$$

3. Fixed point results

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metric space by Perov in 1964 [27], Precup [26]. Let us recall now some useful definitions and results.

Definition 3.1. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc. (i.e.$|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M-\lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$).*

Definition 3.2. *We say that a non-singular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the absolute value property if*

$$A^{-1}|A| \leq I,$$

where

$$|A| = (|a_{ij}|)_{1\leq i,j\leq n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Lemma 3.1. [20] *Let M be a square matrix of nonnegative numbers. The following assertions are equivalent:*

(i) *M is convergent towards zero;*

(ii) *the matrix $I - M$ is non-singular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iii) $\|\lambda\| < 1$ *for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$*

(iv) *$(I - M)$ is non-singular and $(I - M)^{-1}$ has nonnegative elements;*

Some examples of matrices convergent to zero are the following:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1,\ c < 1$

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1,\ a > 1, b > 0$.

For other examples and considerations on matrices which converge to zero, see Precup [26], Rus [40], and Turinici [39].

We can recall now a fixed point theorem in a complete generalized metric space.

Theorem 3.1. [27]*Let (X,d) be a complete generalized metric space with $d: X\times X \longrightarrow \mathbb{R}^n$ and let $N: X \longrightarrow X$ be such that*

$$d(N(x),N(y)) \leq Md(x,y)$$

for all $x,y\in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \longrightarrow 0$ as $k\longrightarrow\infty$, then N has a unique fixed point $x_*\in X$

$$d(N^k(x_0),x_*) \leq M^k(I-M)^{-1}d(N(x_0),x_0)$$

for every $x_0\in X$ and $k\geq 1$.

We suppose that $0\in\rho(A_i)$ (the resolvent set of A_i, for each $i=1,2$).that the semigroup $S_i(t)$ is uniformly bounded, that is to say, $\|S_i(t)\|\leq \bar{M}_1$, for some constant $\bar{M}_1\geq 1$ and for every $t\geq 0$. For $0<\alpha\leq 1$,it is possible to define the fractional power operator $(-A_i)^\alpha$ as a closed linear operator on its domain $D((-A_i)^\alpha)$ with inverse $(-A_i)^{-\alpha}$. Furthermore, the sub-space $D((-A_i)^\alpha)$ is dense in X. We denote by X_α the Banach space $D((-A_i)^\alpha)$ endowed with the norm $\|x\|_\alpha = \|(-A_i)^\alpha x\|$ for $x\in D((-A_i)^\alpha)$ defines a norm on $D((-A_i)^\alpha)$, which is equivalent to the graph norm of $(-A_i)^\alpha$, we represent X_α the space $D((-A_i)^\alpha)$ with the norm $\|\cdot\|_\alpha$. then the following properties are well known (cf. Pazy. ([37]), p. 74).

Lemma 3.2. **(A):** *If $0<\beta<\alpha\leq 1$, then $X_\alpha\subset X_\beta$ and the embedding is compact whenever the resolvent operator of A_i is compact.*

(B): *For each $0<\alpha\leq 1$, there exists a positive constant C_α such that* $\|(-A_i)^\alpha S_i(t)\| \leq \frac{C_\alpha}{t^\alpha}e^{-\lambda t},\ t>0,\ \lambda>0.$

We are now in a position to state and prove our local existence result for the problem (1.1). First we will list the following hypotheses which will be imposed in our main theorem.

- (H1) A_i is the infinitesimal generator of an analytic semigroup, $S_i(t)$ of bounded linear operators on X. Further, to avoid unnecessary notations, we suppose that $0\in\rho(A_i)$, and that, see Lemma 3.2, and there exists a constant M such that $\{\|S_i(t)\|^2\leq M\}$ for all $t\geq 0$

$$\|(-A_i)^{1-\beta}S_i(t)\| \leq C_{1-\beta}t^{\beta-1}$$

for some constants $M, C_{1-\beta}$ and every $t\in[0,b]$.

- (H2)

(i): There exist constants $0 < \beta < 1$, $L_{g_{i1}} \geq 0$ and g^i is X_β-valued, $(-A_i)^\beta g^i$ is continuous, and

$$\|(-A_i)^\beta g^i(t, y_1, y_2)\|^2 \leq L_{g_{i1}}(1 + \|y_1\|^2 + \|y_2\|^2),\ t \in J,\ y_1, y_2 \in X$$

(ii): There exist constants $0 < \beta < 1$, $L_{g_i}, L_{\bar{g}_i} \geq 0$, and

$$\|(-A_i)^\beta g^i(t, x, y) - (-A_i)^\beta g^i(t, \bar{x}, \bar{y})\| \leq L_{g_i}\|x - \bar{x}\| + L_{\bar{g}_i}\|y - \bar{y}\|,\ t \in J,$$

$x, y, \bar{x}, \bar{y} \in X$

- (H3) The map $f^i : [0, \infty) \times X \times X \to X$ satisfies the following condition: for all $t \geq 0$, $x, y, \bar{x}, \bar{y} \in X$ that is, there exist positive constants $L_{f_i}, L_{\bar{f}_i}$ and $L_{f_{i1}}$,$i = 1, 2$ such that,

$$\|f^i(t, x, y) - f^i(t, \bar{x}, \bar{y})\| \leq L_{f_i}\|x - \bar{x}\| + L_{\bar{f}_i}\|y - \bar{y}\|,$$

and

$$\|f^i(t, x, y)\|^2 \leq L_{f_{i1}}(1 + \|x\|^2 + \|y\|^2),$$

- (H4) There existe a constant $c_i, \bar{c}_i$ for each $i = 1, 2$ such that

$$\|I^i_k(x, y) - I^i_k(\bar{x}, \bar{y})\| \leq c_i\|x - \bar{x}\| + \bar{c}_i\|y - \bar{y}\|,$$

for all $x, \bar{x},\ y, \bar{y} \in X$ and $t \in J$.

- (H5)The function $\sigma^i : J \longrightarrow L^0_{Q_i}(Y, X)$ satisfies

$$\int_0^b \|\sigma^i(s)\|^2_{L^0_{Q_i}}\, ds < \infty.$$

- (H6) There exists a positive constant $L_{h_{i1}}, L_{\bar{h}_{i1}}$,$i = 1, 2$ such that,

$$\int_{\mathcal{Z}} \|h^i(s, x, y, \kappa) - h^i(s, \bar{x}, \bar{y}, \kappa)\|^2 \nu(d\kappa) \leq L_{h_{i1}}\|x - \bar{x}\|^2 + L_{\bar{h}_{i1}}\|y - \bar{y}\|^2$$

and

$$\int_{\mathcal{Z}} \|h^1(s, x, y, \kappa)\|^2 \nu(d\kappa) \leq L_{h_{i1}}(1 + \|x\|^2 + \|y\|^2)$$

for all $x, \bar{x},\ y, \bar{y} \in X$ and $t \in J$.

Now, we first define the concept of mild solution to our problem.

Definition 3.3. *Aa $X-$ valued stochastic process $u = (x, y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$ is called a mild solution of the problem* (1.1) *with respect to the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*,if:*

1) *$u(t)$ is $\mathcal{F}_t$-adapted for all $t \in J_k = (t_k, t_{k+1}]$ $\quad k = 1, 2, \ldots, m$;*
2) *$u(t)$ is right continuous and has limit on the left almost surely;*

3) *$u(t)$ satisfies for all $t \in [-\tau, b]$ and almost surely that,*

$$
\begin{cases}
x(t) &= S_1(t)(\phi_1(0) + g^1(0,\phi_1(-u(0)),\phi_2(-u(0))) - g^1(t, x(t-u(t)), y(t-u(t))) \\
&- \int_0^t A_1 S_1(t-s) g^1(s, x(s-u(s)), y(s-u(s)))ds \\
&+ \int_0^t S_1(t-s) f^1(s, x(s-r(s)), y(s-r(s)))ds + \int_0^t S(t-s)\sigma^1(s)dZ_{Q_1}(s) \\
&+ \int_0^t S_1(t-s) \int_{\mathcal{Z}} h^1(s, x(s-\rho(s)), y(s-\rho(s)), \kappa)\widetilde{N}(ds, d\kappa) \\
&+ \sum_{0<t<t_k} S_1(t-t_k) I_k^1(x(t_k), y(t_k)) \\
\\
y(t) &= S_2(t)(\phi_2(0) + g^2(0,\phi_1(-u(0)),\phi_2(-u(0))) - g^2(t, x(t-u(t)), y(t-u(t))) \\
&- \int_0^t A_2 S_2(t-s) g^2(s, x(s-u(s)), y(s-u(s)))ds \\
&+ \int_0^t S_2(t-s) f^2(s, x(s-r(s)), y(s-r(s)))ds + \int_0^t S(t-s)\sigma^2(s)dZ_{Q_2}(s) \\
&+ \int_0^t S_2(t-s) \int_{\mathcal{Z}} h^2(s, x(s-\rho(s)), y(s-\rho(s)), \kappa)\widetilde{N}(ds, d\kappa) \\
&+ \sum_{0<t<t_k} S_2(t-t_k) I_k^2(x(t_k), y(t_k))
\end{cases}
\tag{3.8}
$$

4. Existence and uniqueness of the mild solution

Theorem 4.1. *Suppose that $(H1)-(H6)$ hold and that. Then, problem (1.1) possesses a unique mild solution on $[-\tau, b]$.*

Proof. Fix $b > 0$, let $b > 0$, we define

$$
\mathcal{D}_{\mathcal{F}_b} = \big\{x\colon [-\tau, b]\times\Omega \to X,\ x_k \in C(J_k, X) \text{ for } k = 1,\dots m,\ \phi_i \in \mathcal{D}_{\mathcal{F}_0},\ \text{and there exist } x(t_k^-) \text{ and } x(t_k^+) \text{ with } x(t_k) = x(t_k^-),\ \ k = 1,\cdots,m, \text{ and } \mathbb{E}(\sup_{t\in[0,b]} \|y(t)\|^2) < \infty\big\},
$$

endowed with the norm

$$
\|x\|_{\mathcal{D}_{\mathcal{F}_b}} = \mathbb{E}(\sup_{0\le s\le b} \|x(s)\|^2)^{\frac{1}{2}},
$$

and

$$
S_b(\phi) := \Big\{x \in \mathcal{D}_{\mathcal{F}_T},\ \ x(s) = \phi(s),\ \text{ for }\ s \in [-\tau, 0]\Big\}
$$

Then, $S_b(\phi_1)$ is a closed subset of $\mathcal{D}_{\mathcal{F}_b}$ with the norm $\|x\|_{\mathcal{D}_{\mathcal{F}_b}}$. Consider the operator $N : S_b(\phi)\times S_b(\phi) \to S_b(\phi)\times S_b(\phi)$ defined by

$$
N(x, y) = (N_1(x, y), N_2(x, y)),\ (x, y) \in S_b(\phi)\times S_b(\phi)
$$

where

$$N_1(x(t),y(t)) = \begin{cases} \phi_1(t) \quad t \in [-\tau, 0] \\ \\ S_1(t)(\phi_1(0) + g^1(0,\phi_1(-u(0)),\phi_2(-u(0))) - g^1(t,x(t-u(t)),y(t-u(t))) \\ -\displaystyle\int_0^t A_1 S_1(t-s) g^1(s,x(s-u(s)),y(s-u(s)))ds \\ +\displaystyle\int_0^t S_1(t-s) f^1(s,x(s-r(s)),y(s-r(s)))ds + \int_0^t S_1(t-s)\sigma^1(s)dZ_{Q_1}(s) \\ +\displaystyle\int_0^t S_1((t-s)t-s)\int_{\mathcal{Z}} h^1(s,x(s-\rho(s)),y(s-\rho(s)),\kappa)\widetilde{N}(ds,d\kappa) \\ +\displaystyle\sum_{0<t<t_k} S_1(t-t_k) I_k^1(x(t_k),y(t_k)), \quad \mathbb{P}-a.s, \quad t \in J \end{cases}$$

and

$$N_2(x(t),y(t)) = \begin{cases} \phi_2(t) \quad t \in [-\tau, 0] \\ \\ S_2(t)(\phi_2(0) + g^2(0,\phi_1(-u(0)),\phi_2(-u(0))) - g^2(t,x(t-u(t)),y(t-u(t))) \\ -\displaystyle\int_0^t A_2 S(t-s) g^2(s,x(s-u(s)),y(s-u(s)))ds \\ +\displaystyle\int_0^t S_2(t-s) f^2(s,x(s-r(s)),y(s-r(s)))ds + \int_0^t S_2(t-s)\sigma^2(s)dZ_{Q_2}(s) \\ +\displaystyle\int_0^t S_2(t-s)\int_{\mathcal{Z}} h^2(s,x(s-\rho(s)),y(s-\rho(s)),\kappa)\widetilde{N}(ds,d\kappa) \\ +\displaystyle\sum_{0<t<t_k} S_2(t-t_k) I_k^2(x(t_k),y(t_k)), \quad \mathbb{P}-a.s, \quad t \in J \end{cases}$$

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator N.

Now, we aim to prove that the operator N has a fixed point by means of the Perov's fixed point theorem. The proof will be divided into the following two steps.

Step 1. Next we show that $N(x,y)(t) = (N_1(x,y)(t), N_2(x,y)(t))$ is càdlàg process on $S_T(\phi)$. For arbitrary $(x,y) \in S_b(\phi) \times S_b(\phi)$, we will prove that $t \to N(x,y)(t)$ is continuous on the interval $[0,b]$ in the $L^2(\Omega, X)$-sense. Let $0 < t < b$ and $|h|$ be sufficiently small. Then, for

any fixed $(x,y) \in S_b(\phi) \times S_b(\phi)$, we have

$$
\begin{aligned}
&\|N_1(x,y)(t+h) - N_1(x,y)(t)\| \\
&\qquad \leq \|(S_1(t+h) - S_1(t))(\phi_1(0) + g^1(0, \phi_1(-u(0)), \phi_2(-u(0)))\| \\
&+ \|g^1(t+h, x(t+h-u(t+h)), y(t+h-u(t+h))) - g^1(t, x(t-u(t)), y(t-u(t)))\| \\
&\qquad + \Big\| \int_0^{t+h} A_1 S_1(t+h-s) g^1(s, x(s-u(s)), y(s-u(s))) ds \\
&- \int_0^t A_1 S_1(t-s) g^1(s, x(s-u(s)), y(s-u(s))) ds \Big\| \\
&\qquad + \Big\| \int_0^{t+h} S_1(t+h-s) f^1(s, x(s-r(s)), y(s-r(s))) ds \\
&- \int_0^t S_1(t-s) f^1(s, x(s-r(s)), y(s-r(s))) ds \Big\| \\
&\qquad + \Big\| \int_0^{t+h} S_1(t+h-s) \sigma^1(s) dZ_Q(s) - \int_0^t S_1(t-s) \sigma^1(s) dZ_Q(s) \Big\| \\
&+ \Big\| \int_0^{t+h} S_1(t+h-s) \int_{\mathcal{Z}} h^1(s, x(s-\rho(s)), y(s-\rho(s)), \kappa) \widetilde{N}(ds, d\kappa) \\
&\qquad - \int_0^t S_1(t-s) \int_{\mathcal{Z}} h^1(s, x(s-\rho(s)), y(s-\rho(s)), \kappa) \widetilde{N}(ds, d\kappa) \Big\| \\
&+ \Big\| \sum_{0<t<t_k+h} S_1(t+h-t_k) I_k^1(x(t_k), y(t_k)) - \sum_{0<t<t_k} S_1(t-t_k) I_k^1(x(t_k), y(t_k)) \Big\| := \sum_{l=1}^{7} J_l^1(h)
\end{aligned}
$$

Put

$$
\|N_1(x,y)(t+h) - N_1(x,y)(t)\| = \sum_{l=1}^{7} J_l^1(h) \tag{4.9}
$$

Similar computations for N_2 yield

$$
\|N_2(x,y)(t+h) - N_2(x,y)(t)\| \leq \sum_{l=1}^{7} J_l^2(h) \tag{4.10}
$$

We estimate the various terms of the right hand of (4.9) and (4.10) separately. For the first term, we have

$$
\lim_{h\to 0} (S_1(t+h) - S_1(t))(\phi_1(0) + g^i(0, \phi_1(-u(0)), \phi_2(-u(0))) = 0
$$

and

$$
\begin{aligned}
J_1^1(h) &= \|(S_1(t+h) - S_1(t))(\phi_1(0) + g^1(0, \phi_1(-u(0)), \phi_2(-u(0)))\| \\
&\leq 2M \|\phi_1(0) + g^1(0, \phi_1(-u(0)), \phi_2(-u(0)))\| \in L^2(\Omega)
\end{aligned}
$$

Similarly

$$\begin{aligned}J_1^2(h) &= \|(S_2(t+h)-S_2(t))(\phi_2(0)+g^2(0,\phi_1(-u(0)),\phi_2(-u(0)))\| \\ &\le 2M\|\phi_2(0)+g^2(0,\phi_1(-u(0)),\phi_2(-u(0)))\| \in L^2(\Omega)\end{aligned}$$

Hence, by the Lebesgue dominated theorem, we obtain

$$\lim_{h\to 0}\mathbb{E}\|J_1^i(h)\|^2 = 0, \qquad i=1,2$$

By using assumption $(H2)$ and the fact that the operator $(-A)^{-\beta}$ is bounded, we obtain that

$$\begin{aligned}\mathbb{E}|J_2^i(h)|^2 &= \mathbb{E}\Big\|(-A_i)^{-\beta}(-A_i)^{\beta}\Big(g^i(t+h,x(t+h-u(t+h)),y(t+h-u(t+h))) \\ &\quad - g^i(t,x(t-u(t)),y(t-u(t)))\Big)\Big\|^2 \\ &\le \|(-A_i)^{-\beta}\|^2\mathbb{E}\Big\|(-A_i)^{\beta}\Big(g^i(t+h,x(t+h-u(t+h)),y(t+h-u(t+h))) \\ &\quad - g^i(t,x(t-u(t)),y(t-u(t)))\Big)\Big\|^2 \\ &\to 0, \ \ h\to 0.\end{aligned}$$

To estimate $J_3^i(h)$ for each $i=1,2$. We consider only the case that $h>0$ (for $h<0$ we have the similar estimates hold).

$$\begin{aligned}J_3^i(h) &\le \Big\|\int_0^t A_iS_i(t+h-s)g^i(s,x(s-u(s)),y(s-u(s)))ds \\ &\quad - \int_0^t A_iS_i(t-s)g^i(s,x(s-u(s)),y(s-u(s)))ds\Big\| \\ &\quad + \Big\|\int_t^{t+h} A_iS_i(t+h-s)g^i(s,x(s-u(s)),y(s-u(s)))ds\Big\| \\ &\le \Big\|\int_0^t (S_i(h)-I)(-A_i)^{1-\beta}S_i(t-s)(-A_i)^{-\beta}g^i(s,x(s-u(s)),y(s-u(s)))ds\Big\| \\ &\quad + \Big\|\int_0^t (-A_i)^{1-\beta}S_i(t+h-s)(-A_i)^{-\beta}g^i(s,x(s-u(s)),y(s-u(s)))ds\Big\| \\ &:= J_{31}^i(h)+J_{32}^i(h).\end{aligned}$$

For the term $J_{31}^i(h)$. We have

$$\mathbb{E}|J_{31}^i(h)|^2 \le t\mathbb{E}\int_0^t \|(S_i(h)-I)(-A_i)^{1-\beta}S_i(t-s)(-A_i)^{-\beta}g^i(s,x(s-u(s)),y(s-u(s)))\|^2ds$$

and

$$\lim_{h\to 0}\|(S_i(h)-I)(-A_i)^{1-\beta}S_i(t-s)(-A_i)^{-\beta}g^i(s,x(s-u(s)),y(s-u(s)))\| = 0$$

since the strong continuity of $S(t)$. By conditions $(H1)$ and $(H2)$, we have

$$\begin{aligned}
J_{31}^1(h) =&\|(S_1(h)-I)(-A_1)^{1-\beta}S_1(t-s)(-A_1)^{-\beta}g^1(s,x(s-u(s)),y(s-u(s)))\|^2\\
\le& \frac{4M^2M_{1-\beta}^2}{(t-s)^{2-2\beta}}\|(-A_1)^{-\beta}g^1(s,x(s-u(s)),y(s-u(s)))\|^2\\
\le& L_{g_{11}}\frac{4M^2M_{1-\beta}^2}{(t-s)^{2-2\beta}}(1+\|x\|^2+\|y\|^2).
\end{aligned}$$

Similarly

$$J_{31}^2(h) \le L_{g_{12}}\frac{4M^2M_{1-\beta}^2}{(t-s)^{2-2\beta}}(1+\|x\|^2+\|y\|^2).$$

Hence, by the Lebesgue dominated theorem, we have

$$\lim_{h\to 0}\mathbb{E}|J_{31}^i(h)|^2 = 0.$$

Now, we estimate the term $J_{32}(h)$.

$$\begin{aligned}
E|J_{32}^1(h)|^2 \le& \int_t^{t+h}\left\|(-A_1)^{1-\beta}S_1(t+h-s)\right\|^2 ds \times \int_t^{t+h}\left\|(-A_1)^{-\beta}g^1(s,x(s-u(s)),y(s-u(s)))\right\|^2 ds\\
\le& \int_t^{t+h}\frac{M_{1-\beta}^2}{(t+h-s)^{2-2\beta}}ds \times \int_t^{t+h}\left\|(-A_1)^{-\beta}g^1(s,x(s-u(s)),y(s-u(s)))\right\|^2 ds\\
\le& \int_t^{t+h}\frac{M_{1-\beta}^2}{(t+h-s)^{2-2\beta}}ds \times \int_0^b(1+\|x\|^2+\|y\|^2)ds\\
\le& L_{g_{11}}\frac{M_{1-\beta}^2h^{2\beta-1}}{2\beta-1}\int_0^b(1+\|x\|^2+\|y\|^2)ds \to 0,\quad h\to 0.
\end{aligned}$$

Similarly

$$\mathbb{E}|J_{32}^2(h)|^2 \le L_{g_{12}}\frac{M_{1-\beta}^2h^{2\beta-1}}{2\beta-1}\int_0^b(1+\|x\|^2+\|y\|^2)ds \to 0,\quad h\to 0$$

For the term $J_4^i(h)$. We consider also only the case that $h>0$ (for $h<0$ we have the similar estimates hold).

$$\begin{aligned}
J_4^i(h) \le& \Big\|\int_0^t S_i(t+h-s)f^i(s,x(s-r(s)),y(s-r(s)))\\
&- S_i(t-s)f^i(s,x(s-r(s)),y(s-r(s)))ds\Big\|\\
&+\Big\|\int_t^{t+h}S_i(t+h-s)f^i(s,x(s-r(s)),y(s-r(s)))ds\Big\|\\
\le& \Big\|\int_0^t(S_i(h)-I)S_i(t-s)f^i(s,x(s-r(s)),y(s-r(s)))ds\Big\|\\
&+\Big\|\int_t^{t+h}S_i(t+h-s)f^i(s,x(s-r(s)),y(s-r(s)))ds\Big\|.
\end{aligned}$$

By assumption (H3), we have

$$\begin{aligned}
&\mathbb{E}|J_4^i(h)|^2 \\
&\leq t\int_0^t \Big\|(S_i(h)-I)S_i(t-s)f^i(s,x(s-r(s)),y(s-r(s)))\Big\|^2 ds \\
&+ M^2 h\int_t^{t+h} \Big\|f^i(s,x(s-r(s)),y(s-r(s)))\Big\|^2 ds.
\end{aligned}$$

Noting that

$$\lim_{h\to 0}\Big\|(S_i(h)-I)S_i(t-s)f^i(s,x(s-r(s)),y(s-r(s)))\Big\|^2 = 0.$$

By conditions (H1) and (H2), we have

$$\begin{aligned}
\Big\|((S_1(h)-I)S_1(t-s)f^1(s,x(s-r(s)),y(s-r(s)))\Big\|^2 &\leq 4M^4\Big\|f^1(s,x(s-r(s)),y(s-r(s)))\Big\|^2 \\
&\leq 4L_{f_{11}}M^4(1+\|x\|^2+\|y\|^2)
\end{aligned}$$

Similarly for $t\in[0,b]$, we have the estimate

$$\Big\|((S_2(h)-I)S_2(t-s)f^2(s,x(s-r(s)),y(s-r(s)))\Big\|^2 \leq 4L_{f_{12}}M^4(1+\|x\|^2+\|y\|^2).$$

Hence, by the Lebesgue dominated theorem, we have

$$\lim_{h\to 0}\mathbb{E}|J_4^i(h)|^2 = 0.$$

For the term $J_5^i(h)$, see details [38] . By condition (H5), Lemma 2.1 and the Lebesgue dominated theorem , we have

$$\begin{aligned}
&E|J_5^i(h)|^2 \\
&= E\Big\|\int_0^{t+h} S_i(t+h-s)\sigma^i(s)dZ_Q(s) - \int_0^t S_i(t-s)\sigma^i(s)ds\Big\|^2 \\
&\leq E\Big\|\int_0^t (S_i(t+h-s)-S_i(-s)\sigma^i(s)dZ_Q(s)\Big\|^2 + E\Big\|\int_t^{t+h} S_i(t+h-s)\sigma^i(s)dZ_Q(s)\Big\|^2 \\
&\leq C(H)t^{2H-1}\int_0^t \|(S_i(h)-I)S_i(t-s)\sigma^i(s)\|^2 ds + C(H)h^{2H-1}\mathbb{E}\int_t^{t+h}\Big\|S_i(t+h-s)\sigma^i(s)\Big\|^2 ds \\
&\leq C(H)b^{2H-1}M^2\int_0^t \|(S_i(h)-I)\sigma^i(s)\|^2 ds + C(H)h^{2H-1}\mathbb{E}\int_t^{t+h}\Big\|S_i(t+h-s)\sigma^i(s)\Big\|^2 ds \\
&\to 0,\ \ h\to 0.
\end{aligned}$$

Since

$$\lim_{h\to 0}\|(S_i(h)-I)S_i(t-s)\sigma^1(s)\|^2 = 0$$

and

$$\|(S_i(h)-I)\sigma^i(s)\|^2 \le 2M^2\|\sigma^i(s)\|^2 < \infty$$

$$\|(S_i(h)-I)\sigma^i(s)\|^2 \le M^2\|\sigma^i(s)\|^2 < \infty.$$

The condition (H4) assures that

$$\mathbb{E}|J_6^i(h)|^2 \le M^2 \sum_{0<t_k<t} \Big\|(S_i(h)-I)I_k^i(x(t_k),y(t_k))\Big\|^2 + \sum_{t<t_k<t+h} \Big\|S_i(t+h-t_k)I_k^i(x(t_k),y(t_k))\Big\|^2.$$

Noting that

$$\lim_{h\to 0}\Big\|(S_i(h)-I)S_i(t-t_k)I_k^i(x(t_k),y(t_k))\Big\|^2 = 0.$$

By assumption (H6), we have

$$\begin{aligned}
&\mathbb{E}|J_7^i(h)|^2\\
&= 2\mathbb{E}\Big\|\int_0^t (S_i(t+h-s)-S_i(t-s))\int_{\mathcal{Z}} h^i(s,x(s-\rho(s)),y(s-\rho(s)),\kappa)\widetilde{N}(ds,d\kappa)\Big\|^2\\
&+ 2\mathbb{E}\Big\|\int_t^{t+h} S_i(t+h-s)\int_{\mathcal{Z}} h^i(s,x(s-\rho(s)),y(s-\rho(s)),\kappa)\widetilde{N}(ds,d\kappa)\Big\|^2\\
&\le 2M^2\|S_i(h)-I\|^2 E\mathbb{E}\int_0^t\int_{\mathcal{Z}}\|h^i(s,x(s-\rho(s)),y(s-\rho(s)),\kappa)\|^2\nu(d\kappa)ds\\
&\le 2M^2\mathbb{E}\int_t^{t+h}\int_{\mathcal{Z}}\|h^i(s,x(s-\rho(s)),y(s-\rho(s)),\kappa)\|^2\nu(d\kappa)ds,
\end{aligned}$$

and

$$\int_0^t\int_{\mathcal{Z}}\|h^1(s,x(s-\rho(s)),y(s-\rho(s)),\kappa)\|^2\nu(d\kappa)ds \le L_{h_{11}}t(1+\|x\|^2+\|y\|^2) < \infty.$$

Using the strong continuity of $S_i(t)$ and the Lebesgue dominated theorem , we get

$$\lim_{h\to 0}\mathbb{E}|J_7^i(h)|^2 = 0$$

The above arguments show that

$$\|N(x,y)(t+h)-N(\bar{x},\bar{y})(t)\| \;=\; \begin{pmatrix}\|N_1(x,y)(t+h)-N_1(\bar{x},\bar{y})(t)\|\\ \|N_2(x,y)(t+h)-N_2(\bar{x},\bar{y})(t)\|\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \text{ as } h\to 0.$$

The above arguments show that $N(x,y)(t)$ is càdlàg process. Then, we conclude that $N(S_b(\phi)\times S_b(\phi)) \subset S_b(\phi)\times S_b(\phi)$

Step 2.Now, we are going to show that $N : S_b(\phi) \times S_b(\phi) \to S_b(\phi) \times S_b(\phi)$ is a contraction mapping. For this end, fixe $x, y \in S_b(\phi) \times S_b(\phi)$, we have

$$
\begin{aligned}
&\|N_1(x,y)(t) - N_1(\bar{x},\bar{y})(t)\|^2 \\
&\leq 5\|(-A_1)^{-\beta}\|^2 \Big\|(-A_1)^{\beta}(g^1(t, x(t-u(t)), y(t-u(t))) - g^1(t, \bar{x}(t-u(t)), \bar{y}(t-u(t))))\Big\|^2 \\
&+ 5\Big\|\int_0^t (-A_1)^{1-\beta} S_1(t-s)(-A_1)^{\beta}(g^1(s, x(s-u(s)), y(s-u(s))) - g^1(s, \bar{x}(s-u(s)), \bar{y}(s-u(s))))ds\Big\|^2 \\
&+ 5\Big\|\int_0^t S_1(t-s)(f^1(s, x(s-r(s)), y(s-r(s))) - f^1(s, \bar{x}(s-r(s)), \bar{y}(s-r(s))))ds\Big\|^2 \\
&+ 5\Big\|\int_0^t S_1(t-s)\int_{\mathcal{Z}} h^1(s, x(s-\rho(s)), y(s-\rho(s)), \kappa) - h^1(s, \bar{x}(s-\rho(s)), \bar{y}(s-\rho(s)), \kappa)\widetilde{N}(ds, d\kappa)\Big\|^2 \\
&+ 5\Big\|\sum_{0<t<t_k} S_1(t-t_k)(I_k^1(x(t_k), y(t_k)) - I_k^1(\bar{x}(t_k), \bar{y}(t_k)))\Big\|^2.
\end{aligned}
$$

From assumption (H_1)-(H_6) and Lemma 2.1, yields that,

$$
\begin{aligned}
&\mathbb{E}\|N_1(x,y)(t) - N_1(\bar{x},\bar{y})(t)\|^2 \\
&\leq 5\|(-A_1)^{-\beta}\|^2 \Big(L_{g_1}^2 E\|x(t-u(t)) - \bar{x}(t-u(t))\|^2 + L_{\bar{g}_1}^2 E\|y(t-u(t)) - \bar{y}(t-u(t))\|^2\Big) \\
&+ 5M_{1-\beta}^2 \frac{t^{2\beta-1}}{2\beta-1}\Big(L_{g_1}^2 \int_0^t E\|x(s-u(s)) - \bar{x}(s-u(s))\|^2 ds + L_{\bar{g}_1}^2 \int_0^t E\|y(s-u(s)) - \bar{y}(s-u(s))\|^2 ds\Big) \\
&+ 5tM^2\Big(L_{f_1}^2 \int_0^t \mathbb{E}\|x(s-u(s)) - \bar{x}(s-u(s))\|^2 ds + L_{\bar{f}_1}^2 \int_0^t \mathbb{E}\|y(s-u(s)) - \bar{y}(s-u(s))\|^2 ds\Big) \\
&+ 5tM^2\Big(L_{h_1}^2 \int_0^t \mathbb{E}\|x(s-u(s)) - \bar{x}(s-u(s))\|^2 ds + L_{\bar{h}_1}^2 \int_0^t \mathbb{E}\|y(s-u(s)) - \bar{y}(s-u(s))\|^2 ds\Big) \\
&+ 5M^2\Big(c_1\mathbb{E}\|x(t) - \bar{x}(t)\|^2 + \bar{c}_1 E\mathbb{E}\|y(t) - \bar{y}(t)\|^2\Big).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\mathbb{E}(\sup_{s\in[-\tau,t]}\|N_1(x,y)(s)-N_1(\bar{x},\bar{y})(s)\|^2)\\
&\leq \int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}e^{-\tau\widehat{\alpha}(s)}\mathbb{E}(\sup_{s\in[-\tau,t]}\|x(s)-\overline{x}(s)\|^2)ds+5M^2c_1E(\sup_{t\geq 0}\|x(t)-\bar{x}(t)\|^2)\\
&+\int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}e^{-\tau\widehat{\alpha}(s)}\mathbb{E}(\sup_{s\in[-\tau,t]}\|y(s)-\overline{y}(s)\|^2)ds+5M^2\bar{c}_1E(\sup_{t\geq 0}\|y(t)-\bar{y}(t)\|^2)\\
&\leq \int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}ds\|x-\overline{x}\|_*^2+\int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}ds\|y-\overline{y}\|_*^2\\
&+\bar{C}_1\Big(E(\sup_{t\geq 0}\|x(t)-\bar{x}(t)\|^2)+\mathbb{E}(\sup_{t\geq 0}\|y(t)-\bar{y}(t)\|^2)\Big)\\
&\leq \frac{1}{\tau}\int_0^t (e^{\tau\widehat{\alpha}(s)})'ds\|x-\overline{x}\|_*^2+\frac{1}{\tau}\int_0^t (e^{\tau\widehat{\alpha}(s)})'ds\|y-\overline{y}\|_*^2\\
&+e^{\tau\widehat{\alpha}(t)}e^{-\tau\widehat{\alpha}(t)}\bar{C}_1\Big(\mathbb{E}(\sup_{t\geq 0}\|x(t)-\bar{x}(t)\|^2)+E(\sup_{t\geq 0}\|y(t)-\bar{y}(t)\|^2\Big)\\
&\leq (\frac{1}{\tau}+\bar{C}_1)e^{\tau\widehat{\alpha}(t)}\|x-\overline{x}\|_*^2+(\frac{1}{\tau}+\bar{C}_1)e^{\tau\widehat{\alpha}(t)}\|y-\overline{y}\|_*^2,
\end{aligned}
$$

where

$$\bar{C}_1=\max\{4M^2c_1,4M^2\bar{c}_1\},$$

and

$$\gamma_i(t)=5\|(-A_i)^{-\beta}\|^2L_{g_i}^2+5M_{1-\beta}^2\frac{t^{2\beta}}{2\beta-1}L_{g_i}^2+5t^2M^2L_{f_i}^2+5M^2c_i$$

and

$$\bar{\gamma}_i(t)=5\|(-A_i)^{-\beta}\|^2L_{\bar{g}_i}^2+5M_{1-\beta}^2\frac{t^{2\beta}}{2\beta-1}L_{\bar{g}_i}^2+5t^2M^2L_{\bar{f}_i}^2+5M^2\bar{c}_i.$$

Therefore

$$e^{-\tau\widehat{\alpha}(t)}\mathbb{E}\Big(\sup_{t\in[-\tau,b]}\|N_1(x(t),y(t))-N_1(\overline{x}(t),\overline{y}(t))\|^2\Big)\quad\leq\quad(\tfrac{1}{\tau}+\bar{C}_1)\Big[\|x-\overline{x}\|_*^2+\|y-\overline{y}\|_*^2\Big],$$

where $\|\cdot\|_*$ is the Bielecki-type norm on $S_b(\phi)$ defined by

$$\|x\|_*^2=\mathbb{E}(\sup_{t\in[0,b]}\|x(t,.)\|^2)e^{-\tau\widehat{\alpha}(t)}$$

where

$$\widehat{\alpha}(t)=\int_0^t\alpha(s)ds,\quad t\in[0,b],$$

and

$$\alpha(s)=\max\{\gamma_i(t),\bar{\gamma}_i(t)\}.$$

Hence

$$\|N_1(x,y)-N_1(\overline{x},\overline{y})\|_*^2\quad\leq\quad(\tfrac{1}{\tau}+C_1)\|x-\overline{x}\|_*^2+(\tfrac{1}{\tau}+\bar{C}_1)\|y-\overline{y}\|_*^2.$$

Using the fact that for all $a, b \geq 0$ we have $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we conclude that

$$\|N_1(x,y) - N_1(\overline{x},\overline{y})\|_* \leq \sqrt{\tfrac{1}{\tau} + \bar{C}_1}\|x - \overline{x}\|_* + \sqrt{\tfrac{1}{\tau} + \bar{C}_1}\|y - \overline{y}\|_*.$$

Similar computations for N_1 yield

$$\|N_2(x,y) - N_2(\overline{x},\overline{y})\|_* \leq \sqrt{\tfrac{1}{\tau} + \bar{C}_2}\|x - \overline{x}\|_* + \sqrt{\tfrac{1}{\tau} + \bar{C}_2}\|y - \overline{y}\|_*.$$

where

$$\bar{C}_2 = \max\{4M^2 c_2, 4M^2 \bar{c}_2\}, \ \ \tau' = \max\left\{\frac{\tau}{1+\tau C_1}, \frac{\tau}{1+\tau C_2}\right\}.$$

Thus

$$\begin{aligned} \|N(x,y) - N(\overline{x},\overline{y})\|_* &= \begin{pmatrix} \|N_1((x,y) - N_1(\overline{x},\overline{y})\|_* \\ \|N_2(x,y) - N_2(\overline{x},\overline{y})\|_* \end{pmatrix} \\ &\leq \frac{1}{\sqrt{\tau'}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \|x - \overline{x}\|_* \\ \|y - \overline{y}\|_* \end{pmatrix}. \end{aligned}$$

Hence

$$\|N(x,y) - N(\overline{x},\overline{y})\|_* \leq \frac{1}{\sqrt{\tau'}} M_{\alpha,\beta}\begin{pmatrix} \|x - \overline{x}\|_* \\ \|y - \overline{y}\|_* \end{pmatrix},$$

for all $(x,y), (\overline{x},\overline{y}) \in S_b(\phi) \times S_b(\phi)$, where

$$M_{\alpha,\beta} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

If we choose a suitable $\sqrt{\tau'} > 2$ such that the matrix

$$\frac{\|M_{\alpha,\beta}\|}{\sqrt{\tau'}} < 1,$$

then $\frac{M_{\alpha,\beta}}{\tau'}$ is nonnegative, $I - \frac{\|M_{\alpha,\beta}\|}{\tau'}$ is non singular and

$$\left(I - \frac{M_{\alpha,\beta}}{\sqrt{\tau'}}\right)^{-1} = I + \frac{M_{\alpha,\beta}}{\sqrt{\tau'}} + \frac{M^2_{\alpha,\beta}}{\tau'} + \ldots$$

From Lemma 3.1, we obtain that $\frac{M_{\alpha,\beta}}{\sqrt{\tau'}}$ converges to zero. As a consequence of Perov's fixed point theorem, N has a unique fixed $(x,y) \in S_b(\phi) \times S_b(\phi)$ which is the unique solution of problem (1.1). Let us denote this solution by (x,y).

5. An example

We consider the following impulsive neutral stochastic partial differential equation with Poisson jumps and finite delays driven by a Rosenblatt process of the form:

Example 5.1. *Consider the following couple stochastic partial differential equation with impulsive effects*

$$\begin{cases} d(x(t,\xi) & + & \frac{\alpha_1}{M_{\frac{1}{4}}\|(-A)^{\frac{3}{4}}\|}(x(t-u(t),\xi)+y(t-u(t),\xi)) = \frac{\partial^2}{\partial\xi^2}x(t,\xi) \\ & + & \alpha_2(x(t-r(t),\xi)+y(t-r(t),\xi)) + e^{-t}dZ^H(t) \\ & + & \int_{\mathcal{Z}} \alpha_3\kappa(x(t-\rho(t),\xi)+y(t-\rho(t),\xi)\widetilde{N}(dt,d\kappa), \quad t\geq 0, \quad t\neq t_k, \quad 0\leq\xi\leq\pi, \\ d(y(t,\xi) & + & \frac{\lambda_1}{M_{\frac{1}{4}}\|(-A)^{\frac{3}{4}}\|}(x(t-u(t),\xi)-y(t-u(t),\xi)) = \frac{\partial^2}{\partial\xi^2}x(t,\xi) \\ & + & \lambda_2(x(t-r(t),\xi)-y(t-r(t),\xi)) + e^{-t}dZ^H(t) \\ & + & \int_{\mathcal{Z}} \lambda_3\kappa(x(t-\rho(t),\xi)-y(t-\rho(t),\xi)\widetilde{N}(dt,d\kappa), \quad t\geq 0, \quad t\neq t_k, \quad 0\leq\xi\leq\pi, \\ x(t_k^+,\xi) & - & x(t_k^-,\xi) = \frac{\alpha_4}{2}x(t_k^-,\xi), \quad k=1,\cdots,m, \\ y(t_k^+,\xi) & - & y(t_k^-,\xi) = \frac{\bar{\lambda}_4}{2}y(t_k^-,\xi), \quad k=1,\cdots,m, \\ x(t,0) & = & x(t,\pi)=0, t\geq 0, \;\; \alpha_i,\lambda_i>0, \quad i=1,2,3,4 \\ y(t,0) & = & y(t,\pi)=0, t\geq 0, \\ x(s,\xi) & = & \phi_1(s,\xi), \; 0\leq\xi\leq\pi, \;\; -\tau\leq s\leq 0 \\ y(s,\xi) & = & \phi_2(s,\xi), \; 0\leq\xi\leq\pi, \end{cases} \tag{5.11}$$

Take $Y = X = L^2([0,\pi])$. We define the operator $A_1 = A_2 = A$ by $Au = u''$, with domain $D(A) = \{u \in X, u', u'' \in X \quad \text{and} \quad u(0) = u(\pi) = 0\}$.

Then, it is well known that

$$Ax = \sum_{n=1}^{\infty} n^2 \langle x, e_n\rangle_X e_n, \;\; x \in D(A),$$

and A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on X, which is given by

$S(t)u = \sum_{n=1}^{\infty} e^{-n^2t}\langle u, e_n\rangle e_n, \; u \in X,$ and $e_n(u) = (2/\pi)^{1/2}\sin(nu), n = 1, 2, \cdots,$ is the orthogonal set of eigenvectors of $-A$.

The bounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$(-A)^{\frac{3}{4}}x = \sum_{n=1}^{\infty} n^{\frac{3}{2}}\langle x, e_n\rangle_X e_n,$$

with domain

$$D((-A)^{\frac{3}{4}}) = \{x \in X,\ \sum_{n=1}^{\infty} n^{\frac{3}{2}}\langle x, e_n\rangle_X e_n \in X\}$$

The analytic semigroup $\{S(t)\}_{t>0}$, $t \in J$, is compact, and there exists a constant $M \geq 1$ such that $\|S(t)\|^2 \leq M$.

$Z^H(t)$ is Rosenblatt process with parameter $H \in (1/2, 1)$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In order to define the operator $Q : \mathcal{K} \longrightarrow \mathcal{K}$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$, set $Qe_n = \sigma_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty.$$

Define the process $B_Q^H(s)$ by

$$Z^H(t) = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\gamma_n^H(t)e_n,$$

where $H \in (1/2, 1)$, and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional mutually independent fractional Brownian motions.Now, rewrite (5.11) into the abstract form of (1.1). In order to model the problem (5.11) in the abstract form of (1.1), we consider the mapping f^i, g^i and h^i for each $i = 1, 2$ as follows

$$g^1(t, x(t-u(t)), y(t-u(t))) = \frac{\alpha_1}{M_{\frac{1}{4}}\|(A)^{\frac{3}{4}}\|}(x(t-u(t)) + y(t-u(t))),$$

$$g^2(t, x(t-u(t)), y(t-u(t))) = \frac{\alpha_1}{M_{\frac{1}{4}}\|(A)^{\frac{3}{4}}\|}(x(t-u(t)) - y(t-u(t))),$$

and

$$f^1(t, x(t-u(t)), y(t-u(t))) = \alpha_2(x(t-r(t)) + y(t-r(t))),$$

$$f^2(t, x(t-u(t)), y(t-u(t))) = \lambda_2(x(t-r(t)) - y(t-r(t)))$$

and

$$h^1(t, x(t-\rho(t)), y(t-\rho(t)), \kappa) = \alpha_3\kappa(x(t-\rho(t)) + y(t-\rho(t)),$$

and

$$h^2(t, x(t-\rho(t)), y(t-\rho(t)), \kappa) = \lambda_3\kappa(x(t-\rho(t)) - y(t-\rho(t))$$

More precisely, f^i, g^i and h^i satisfy Lipschitz condition with $\|(-A)^{\frac{3}{4}}\| = 1$ $L_{f_1} = L_{\bar{f}_1} = \alpha_2, L_{f_2} = L_{\bar{f}_2} = \lambda_2$ and $L_{g_1} = L_{\bar{g}_1} = \alpha_2, L_{g_2} = L_{\bar{g}_2} = \frac{\alpha_1}{M_{\frac{1}{4}}}$, $L_{h_1} = L_{\bar{h}_1} = \int_{\mathcal{Z}} \alpha_3^2\kappa^2\nu(d\kappa)$ and $c_1 = \bar{c}_1 = \frac{\alpha_4}{2}$, $c_2 = \bar{c}_2 = \frac{\lambda_4}{2}$.Thanks to these assumptions, it is straightforward to check

that $(H1)-(H6)$ *hold true and, then, assumptions in Theorem 4.1 are fulfilled, and we can conclude that system (5.11) possesses a mild solution on* $[-\tau, b]$.

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NONEXISTENCE OF GLOBAL SOLUTIONS TO SEMI-LINEAR FRACTIONAL EVOLUTION EQUATION

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ABSTRACT. In this paper, we consider the following semi-linear fractional evolution equation

$$u_{tt} + (-\Delta)^{\frac{\beta}{2}} u + D^{\alpha}_{0|t} u = h(t,x) \left|u\right|^p ,$$

posed in $(0,T) \times \mathbb{R}^N$, where $(-\Delta)^{\frac{\beta}{2}}$, $0 < \beta \leq 2$ is $\frac{\beta}{2}-$ fractional power of $-\Delta$, and $D^{\alpha}_{0/t}$ denotes the derivatives of order α in the sense of Caputo. The nonexistence of global solutions theorem is established. Our method of proof is based on suitable choices of the test functions in the weak formulation of the sought solutions.

1. Introduction

Our main interest lies in the following problem:

$$\begin{cases} u_{tt} + (-\Delta)^{\frac{\beta}{2}} u + D^{\alpha}_{0|t} u = h(t,x) \left|u\right|^p , & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ \\ u(0,x) = u_0(x) \geq 0,, \ u_t(0,x) = u_1(x) \geq 0,, \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p > 1, 0 < \alpha < 1, 0 < \beta \leq 2$ are constants. The function h is a non-negative and assumed to satisfy the condition

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$$h(t,x) \geq Ct^{\nu} |x|^{\mu}, \text{where } \ C>0,\ \nu \geq 0,\ \mu \geq 0. \tag{1.2}$$

$D^{\alpha}_{0/t}$ denotes the derivatives of order α in the sense of Caputo and $(-\Delta)^{\frac{\beta}{2}}$ is the fractional power of $(-\Delta)$. The integral representation of the fractional Laplacian in the N-dimensional space [17] is

$$(-\Delta)^{\beta/2}\psi(x) = -c_N(\beta)\int_{\mathbb{R}^N} \frac{\psi(x+z)-\psi(x)}{|z|^{N+\beta}}dz, \quad \forall x \in \mathbb{R}^N, \tag{1.3}$$

where $c_N(\beta) = \Gamma((N+\beta)/2)/(2\pi^{N/2+\beta}\Gamma(1-\beta/2))$ and Γ denotes the gamma function. Note that the fractional Laplacian $(-\Delta)^{\beta/2}$ with $\beta \in (0,2]$ is a pseudo-differential operator defined by:

$$(-\Delta)^{\beta/2}u(x) = \mathcal{F}^{-1}\big[\, |\zeta|^{\beta}\, \mathcal{F}(u)(\zeta)\big](x) \ \text{ for all } \quad x \in \mathbb{R}^N,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier transform and its inverse, respectively. Let us point out that many authors investigated the cases where $\alpha = 1, \beta = 2$ in several contexts. For example, the following Cauchy problem:

$$\begin{cases} u_{tt} + u_t - \Delta u = |u|^p, & (t,x) \in (0,\infty) \times (\mathbb{R}^N) \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

has been investigated by Qi. Zhang [20] in the case $1 < p < 1 + \frac{2}{N}$, when $u_i, i = 0,1$ is compactly supported and $\int u_i(x)dx > 0$. He proved that the solution of (1.4) does not exist globally. Therefore, he showed that $p = 1 + \frac{2}{N}$ belongs to the blow-up case.

Ogawa and H. Takeda [13] studied (1.4) as a initial boundary value problem in an exterior domain Ω. They established the non-existence of non-negative global solutions of the above problem when $1 < p < 1 + \frac{2}{N}$ and the initial data $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ having a compact support. After that Fino and Wehbe [3] generalized the results of Ogawa-Takeda [13] by proving the blow-up of solutions of (1.4) under weaker assumptions on the initial data and they extended this results to the critical case $p = 1 + \frac{2}{N}$.

Todorova-Yordanov [19] showed that, if $p_c < p \leq \frac{N}{N-2}$, for $n \geq 3$ and $p_c < p < +\infty$, for $N = 1,2$, where $p_c = 1 + \frac{2}{N}$, then (1.4) subjected to initial data $u(0,x) = \epsilon u_0(x), \quad u_t(0,x) = \epsilon u_1(x),\ \ \epsilon > 0, x \in \mathbb{R}^N$, admits a unique global solution, and they proved that if $1 < p < 1 + \frac{2}{N}$, then the solution u blows up in a finite time. R. Ikehata [10], subjected the problem (1.4) with initial-boundary values, he derived certain decay estimates for the total energy of the

solution to the problem (1.4), when $1+\frac{4}{N+2}<p\leq\frac{N}{N-1}$. In particular, A. Hakem [8] treated the following problem:

$$\begin{cases} u_{tt}+g(t)u_t-\Delta u=|u|^p\,, & (t,x)\in(0,\infty)\times(\mathbb{R}^N), \\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.5}$$

where $g(t)$ is a function behaving like t^β, $0\leq\beta<1$. He obtained the non-existence of weak solution for the problem (1.5), when $1<p\leq\frac{N+2}{N+2\beta}$.
Our purpose of this work is to generalize some of the above results, so with the suitable choice of the test function, we prove the non-existence of nontrivial global weak solution of (1.1).

2. Preliminaries

Set $\Sigma_T=(0,T)\times(\mathbb{R}^N)$. The results of our research are based on the following definitions:

Definition 2.1. *Let $0<\alpha<1$ and $\zeta'\in L^1(0,T)$. The left-sided and respectively right-sided Caputo derivatives of order α for ζ are defined as:*

$$D^\alpha_{0|t}\zeta(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\zeta'(s)}{(t-s)^\alpha}ds \quad \text{and} \quad D^\alpha_{t|T}\zeta(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^T\frac{\zeta'(s)}{(s-t)^\alpha}ds,$$

where Γ denotes the gamma function (see [14] p 79).

Definition 2.2. *We say that $u\geq0$ is a local weak solution to (1.1), defined in Σ_T, $0<T<+\infty$, if u is a locally integrable function such that $u^ph\in L^1_{loc}(\Sigma_T)$ and*

$$\int_{\Sigma_T}h\,|u|^p\,\Psi\,dx\,dt+\int_{\mathbb{R}^N}u_0(x)\Psi(0,x)\,dx+\int_{\mathbb{R}^N}u_1(x)\Psi(0,x)\,dx-\int_{\mathbb{R}^N}u_0(x)\Psi_t(0,x)\,dx$$
$$=\int_{\Sigma_T}u\Psi_{tt}\,dx\,dt+\int_{\Sigma_T}uD^\alpha_{t|T}\Psi\,dx\,dt+\int_{\Sigma_T}u(-\Delta)^{\frac{\beta}{2}}\Psi\,dx\,dt,$$

is satisfied for any $\Psi\in C^{2,2}_{t,x}\Big(\Sigma_T\Big)$ such that $\Psi(T,.)=\Psi_t(T,.)=0$.

Definition 2.3. *We say that $u\geq0$ is global weak solution to* (1.1) *if it is a local solution to* (1.1) *defined in Σ_T for any $T>0$.*

Now, we recall the following integration by parts formula (see [18] p 46):

$$\int_0^T\phi(t)(D^\alpha_{0|t}\psi)(t)dt=\int_0^T(D^\alpha_{t|T}\phi)(t)\psi(t)dt. \tag{2.6}$$

We notice that, in all steps of proof, $C>0$ is a real positive number which may change from line to line.

3. Main results

Our main result reads as follows:

Theorem 3.1. *Assume that $p > 1, 0 < \alpha < 1, 0 < \beta \leq 2$ and the conditions (1.2) are satisfied, if*

$$p \leq \frac{\alpha(N+\mu)+\beta(1+\nu)}{\alpha N+\beta(1-\alpha)}, \tag{3.7}$$

then the problem (1.1) *has no nontrivial global weak solutions.*

Proof. Since the principle of the method is the right choice of the test function, we choose it as follows:

$$\Psi(t,x) = \Phi\Big(\frac{t^2+|x|^{\frac{2\beta}{\alpha}}}{R^2}\Big), \quad R>0,$$

where Φ is a cut-off no increasing function satisfying

$$\Phi(r) = \begin{cases} 0, & \text{if } \ r \geq 2, \\ 1, & \text{if } \ r \leq 1, \end{cases}$$

and

$$0 \leq \Phi \leq 1, \quad \text{for all } \ r > 0.$$

Now multiplying the equation (1.1) by Ψ and integrating by parts on $\Sigma_T = (0,T)\times(\mathbb{R}^N)$, we get

$$\begin{aligned} &\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\Psi(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,dx \\ &-\int_{\mathbb{R}^N} u_0(x)\Psi_t(0,x)\,dx = \int_{\Sigma_T} u\Psi_{tt}\,dx\,dt - \int_{\Sigma_T} uD^{\alpha}_{0|t}\Psi\,dx\,dt + \int_{\Sigma_T} u(-\Delta)^{\frac{\beta}{2}}\Psi\,dx\,dt. \end{aligned} \tag{3.8}$$

Invoking the fact that

$$\Psi_t(t,x) = 2tR^{-2}\Phi'\Big(\frac{t^2+|x|^{\frac{2\beta}{\alpha}}}{R^2}\Big),$$

we easily deduce that $\Psi_t(0,x)=0$. By using (2.6), the formula (3.8) will be on the shape

$$\begin{aligned} &\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\Psi(0,x)\,dx + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,dx \\ &= \int_{\Sigma_T} u\Psi_{tt}\,dx\,dt + \int_{\Sigma_T} uD^{\alpha}_{t|T}\Psi\,dx\,dt + \int_{\Sigma_T} u((-\Delta)^{\frac{\beta}{2}}\Psi)\,dx\,dt. \end{aligned} \tag{3.9}$$

To estimate

$$\int_{\Sigma_T} u\Psi_{tt}\,dx\,dt,$$

we observe that

$$\int_{\Sigma_T} u\Psi_{tt}\,dx\,dt = \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}\Psi_{tt}(h\Psi)^{\frac{-1}{p}}\,dx\,dt,$$

we have also

$$\int_{\Sigma_T} uD^{\alpha}_{t|T}\Psi\,dx\,dt=\int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}D^{\alpha}_{t|T}\Psi(h\Psi)^{\frac{-1}{p}}\,dx\,dt,$$

and

$$\int_{\Sigma_T} u((-\Delta)^{\frac{\beta}{2}}\Psi)\,dx\,dt=\int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}((-\Delta)^{\frac{\beta}{2}}\Psi)(h\Psi)^{\frac{-1}{p}}\,dx\,dt.$$

An application of the following ϵ-Young's inequality

$$ab\le \epsilon a^p+C(\epsilon)b^q,\ a>0,\ b>0,\epsilon>0,\ pq=p+q \text{ and } C(\epsilon)=(\epsilon p)^{\frac{-q}{p}}q^{-1},$$

to the first integral of the right hand side of (3.9), we obtain

$$\int_{\Sigma_T} u\Psi_{tt}\,dx\,dt\le \epsilon\int_{\Sigma_T}|u|^p\,h\Psi\,dx\,dt+C(\epsilon)\int_{\Sigma_T}|\Psi_{tt}|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt,$$

and for the second integral of the right hand side of (3.9), we get

$$\int_{\Sigma_T} uD^{\alpha}_{t|T}\Psi\,dx\,dt\le \epsilon\int_{\Sigma_T}|u|^p\,h\Psi\,dx\,dt+C(\epsilon)\int_{\Sigma_T}(\left|D^{\alpha}_{t|T}\Psi\right|)^{\frac{p}{p-1}}(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt.$$

Similarly for the third integral of the right hand side of (3.9), we have

$$\left|\int_{\Sigma_T} u(-\Delta)^{\frac{\beta}{2}}\Psi\,dx\,dt\right|\le \epsilon\int_{\Sigma_T}|u|^p\,h\Psi\,dx\,dt+C(\epsilon)\int_{\Sigma_T}\left|(-\Delta)^{\frac{\beta}{2}}(\Psi)\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt.$$

Finally, we get

$$\begin{aligned}\int_{\Sigma_T}|u|^p\,h\Psi\,dx\,dt\le C\int_{\Sigma_T}|\Psi_{tt}|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt+C\int_{\Sigma_T}\left|D^{\alpha}_{t|T}\Psi\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt\\ +C\int_{\Sigma_T}\left|(-\Delta)^{\frac{\beta}{2}}(\Psi)\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt.\end{aligned}\tag{3.10}$$

By the choice of Ψ, it is easy to show that

$$\begin{cases}\displaystyle\int_{\Sigma_T}|\Psi_{tt}|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt<\infty, \quad \int_{\Sigma_T}\left|D^{\alpha}_{t|T}\Psi\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt<\infty,\\ \\ \displaystyle\int_{\Sigma_T}\left|(-\Delta)^{\frac{\beta}{2}}(\Psi)\right|^{\frac{p}{p-1}}\,(h\Psi)^{\frac{-1}{p-1}}\,dx\,dt<\infty.\end{cases}$$

At this stage, we introduce the scaled variables:

$$\tau=tR^{-1},\ \zeta=xR^{\frac{-\alpha}{\beta}}.$$

Using the fact that

$$dxdt=R^{\frac{N\alpha}{\beta}+1}d\zeta d\tau,\ \Psi_t=R^{-1}\Psi_\tau,\ \Psi_{tt}=R^{-2}\Psi_{\tau\tau},(-\Delta)^{\frac{\beta}{2}}_x\Psi=R^{-\alpha}(-\Delta)^{\frac{\beta}{2}}_\zeta\Psi,\ D^{\alpha}_{t|T}\Psi=R^{-\alpha}D^{\alpha}_{\tau|RT}\Psi,$$

and setting

$$\Omega = \Big\{(\tau,\zeta) \in \mathbb{R}^+ \times \mathbb{R}^N; 1 \le \tau^2 + |\zeta|^{\frac{2\beta}{\alpha}} \le 2\Big\}, \quad \varphi(\tau,\zeta) = \tau^2 + |\zeta|^{\frac{2\beta}{\alpha}},$$

we arrive at

$$\begin{aligned}\int_{\Sigma_T} |u|^p\, h\Psi\, dx\, dt \le{}& CR^{\theta_1} \int_\Omega |(\Psi_{\tau\tau})(\varphi)|^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau \\ &+ CR^{\theta_2} \int_\Omega \Big|(D^\alpha_{\tau|RT})(\varphi)\Big|^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau \\ &+ CR^{\theta_3} \int_\Omega \Big|(-\Delta)^{\frac{\beta}{2}}\Psi(\varphi)\Big|^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau.\end{aligned} \tag{3.11}$$

Where

$$\begin{cases} \theta_1 = \dfrac{N\alpha}{\beta} + 1 - \dfrac{2p}{p-1} - \dfrac{1}{p-1}\Big(\dfrac{\alpha}{\beta}\mu + \nu\Big), \\[2ex] \theta_2 = \dfrac{N\alpha}{\beta} + 1 - \dfrac{\alpha p}{p-1} - \dfrac{1}{p-1}\Big(\dfrac{\alpha}{\beta}\mu + \nu\Big), \\[2ex] \theta_3 = \dfrac{N\alpha}{\beta} + 1 - \dfrac{\alpha p}{p-1} - \dfrac{1}{p-1}\Big(\dfrac{\alpha}{\beta}\mu + \nu\Big). \end{cases}$$

One can easily observe that: $\theta_1 < \theta_2 = \theta_3$, we infer that

$$\begin{aligned}\int_{\Sigma_T} |u|^p\, h\Psi\, dx\, dt \le CR^{\theta}\Bigg[& \int_\Omega |(\Psi_{\tau\tau})(\varphi)|^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau \\ &+ \int_\Omega \Big|(D^\alpha_{\tau|RT})(\varphi)\Big|^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau \\ &+ \int_\Omega \Big|(-\Delta)^{\frac{\alpha}{2}}\Psi(\varphi)\Big|^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, d\zeta\, d\tau\Bigg],\end{aligned} \tag{3.12}$$

where $R > 0$, large and

$$\theta = \theta_2 = \frac{1}{\beta(p-1)}\Big\{[\alpha N + \beta(1-\alpha)]p - \alpha(N+\mu) - \beta(1+\nu)\Big\}.$$

It is clear that $\alpha N + \beta(1-\alpha) > 0$, thus, we distinguish two cases:

- If
$$\theta < 0 \Leftrightarrow p < \frac{\alpha(N+\mu) + \beta(1+\nu)}{\alpha N + \beta(1-\alpha)},$$
then the right-hand side of (3.12) goes to 0 when R tends to infinity, we pass to the limit in the left hand side, as R goes to $+\infty$, we get
$$\lim_{R\to+\infty} \int_{\Sigma_T} h\,|u|^p\, \Psi\, dx\, dt = 0.$$

Using the Lebesgue dominated convergence theorem, the continuity in time and space of u and the fact that $\Psi(t,x) \to 1$ as $R \to +\infty$, we infer that

$$\int_{\mathbb{R}^+ \times \mathbb{R}^N} h\, |u|^p \; dx\, dt = 0.$$

Therefore, if u exists then necessarily $u \equiv 0$ $a.$ $e.$ on $\mathbb{R}^+ \times \mathbb{R}^N$.

- If

$$\theta = 0 \Leftrightarrow p = \frac{\alpha(N+\mu) + \beta(1+\nu)}{\alpha N + \beta(1-\alpha)},$$

then we have

$$\int_{\mathbb{R}^+ \times \mathbb{R}^N} |u|^p \, h \, dx\, dt < +\infty. \tag{3.13}$$

By using (3.9) we obtain

$$\begin{aligned}
&\int_{\Sigma_T} h\,|u|^p\, \Psi\, dx\, dt + \int_{\mathbb{R}^N} u_0(x)\Psi(0,x)\, dx + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\, dx \\
&\leq \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}}\, |\Psi_{tt}|\, (h\Psi)^{\frac{-1}{p}}\, dx\, dt + \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}} \left| D^{\alpha}_{t|T} \right| (h\Psi)^{\frac{-1}{p}}\, dx\, dt \\
&+ \int_{\Sigma_T} u(h\Psi)^{\frac{1}{p}} \left| (-\Delta)^{\frac{\beta}{2}} \Psi \right| (h\Psi)^{\frac{-1}{p}}\, dx\, dt.
\end{aligned} \tag{3.14}$$

Accordingly, using Hölder's inequality in the right hand side of (3.14), yields

$$\begin{aligned}
\int_{\Sigma_T} h\,|u|^p\, \Psi\, dx\, dt &\leq \left(\int_{\Sigma_T} u^p h \Psi\, dx\, dt \right)^{\frac{1}{p}} \left(\int_{\Sigma_T} |\Psi_{tt}|^{\frac{p}{p-1}}\, (h\Psi)^{\frac{-1}{p-1}}\, dx\, dt \right)^{\frac{p-1}{p}} \\
&+ \left(\int_{\Sigma_T} u^p h \Psi\, dx\, dt \right)^{\frac{1}{p}} \left(\int_{\Sigma_T} \left(\left| D^{\alpha}_{t|T} \Psi \right| \right)^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, dx\, dt \right)^{\frac{p-1}{p}} \\
&+ \left(\int_{\Sigma_T} u^p h \Psi\, dx\, dt \right)^{\frac{1}{p}} \left(\int_{\Sigma_T} \left(\left| (-\Delta)^{\frac{\beta}{2}} \Psi \right| \right)^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, dx\, dt \right)^{\frac{p-1}{p}}.
\end{aligned}$$

We easily see that

$$\begin{aligned}
\int_{\Sigma_T} h\,|u|^p\, \Psi\, dx\, dt \leq \left(\int_{\Sigma_T} u^p h \Psi\, dx\, dt \right)^{\frac{1}{p}} \times \Bigg[&\left(\int_{\Sigma_T} |\Psi_{tt}|^{\frac{p}{p-1}}\, (h\Psi)^{\frac{-1}{p-1}}\, dx\, dt \right)^{\frac{p-1}{p}} \\
&+ \left(\int_{\Sigma_T} \left(\left| D^{\alpha}_{t|T} \right| \right)^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, dx\, dt \right)^{\frac{p-1}{p}} \\
&+ \left(\int_{\Sigma_T} \left(\left| (-\Delta)^{\frac{\beta}{2}} \Psi \right| \right)^{\frac{p}{p-1}} (h\Psi)^{\frac{-1}{p-1}}\, dx\, dt \right)^{\frac{p-1}{p}} \Bigg].
\end{aligned}$$

Because $\theta = 0$, we get from (3.13) that

$$\int_{\Sigma_T} h\,|u|^p\,\Psi\,dx\,dt \leq \left(\int_{\Omega_2} u^p h\Psi\,dx\,dt\right)^{\frac{1}{p}} \times \left[\left(\left(\int_{\Omega_1} |\Psi_{\tau\tau}(\varphi)|^{\frac{p}{p-1}}\,(h\Psi(\varphi))^{\frac{-1}{p-1}}\,d\zeta\,d\tau\right)^{\frac{p-1}{p}}\right.\right.$$

$$+\left(\int_{\Omega_1} \left(\left|D^{\alpha}_{\tau|RT}(\varphi)\right|\right)^{\frac{p}{p-1}}(h\Psi(\varphi))^{\frac{-1}{p-1}}\,d\zeta\,d\tau\right)^{\frac{p-1}{p}}$$

$$\left.+\left(\int_{\Omega_1} \left(\left|(-\Delta)^{\frac{\beta}{2}}\Psi(\varphi)\right|\right)^{\frac{p}{p-1}}(h\Psi(\varphi))^{\frac{-1}{p-1}}\,d\zeta\,d\tau\right)^{\frac{p-1}{p}}\right],$$

where

$$\Omega_1 = \left\{(\tau,\zeta) \in \mathbb{R}^+ \times \mathbb{R}^N;\ 1 \leq \tau^2 + |\zeta|^{\frac{2\beta}{\alpha}} \leq 2\right\},$$

and

$$\Omega_2 = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N;\ R^2 \leq t^2 + |x|^{\frac{2\beta}{\alpha}} \leq 2R^2\right\}.$$

Taking into account the fact that $\int_{\mathbb{R}^+\times\mathbb{R}^N} |u|^p\,h\,dx\,dt < +\infty$, we obtain

$$\lim_{R\to+\infty} \int_{\Omega_2} |u|^p\,h\Psi\,dx\,dt = 0,$$

hence, we conclude that

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} |u|^p\,h\,dx\,dt = 0.$$

Whereupon $u \equiv 0$. We deduce that no nontrivial global solution is possible. This finishes the proof.

Remark 3.1. *We observe that in the case $\alpha = 1$, $\beta = 2$, $\mu = \nu = 0$, we retrieve the Fujita's critical exponent $p_c = 1 + \frac{2}{N}$.*

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ON SOME TENSOR CONDITIONS OF NEARLY KENMOTSU f-MANIFOLDS

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ABSTRACT. In this paper, we continue to study on nearly Kenmotsu f-manifolds motivated by previous study. In this time, we prove that a second-order symmetric closed recurrent tensor is a multiple of the associated metric tensor on nearly Kenmotsu f-manifolds. Then, we get some necessary condition under which a vector field on a nearly Kenmotsu f-manifold will be a strict generalized contact or Killing vector field. Finally, we show that every φ-recurrent nearly Kenmotsu f-manifold is an Einstein manifold of globally framed type and every locally φ-recurrent nearly Kenmotsu f-manifold is a manifold of constant curvature.

1. INTRODUCTION

The studies on complex manifold is initiated by Schouten and van Dantzig in 1930 [20]. In 1933, Kähler introduced an important class of complex manifolds, which is called Kähler manifold [13]. Then, Weil proved that the existence of (1, 1) tensor field J on complex manifold, which satisfies

$$J^2 = -I,$$

where I denotes the identity transformation [23]. In 1950, Ehresmann defined almost complex manifolds, using this tensor field J. He proved that every complex manifold is an almost complex manifold, but the converse is not true [7].

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In 1970, A. Gray introduced nearly Kähler manifolds which are not Kähler, using the covariant derivative of almost complex structure J with respect to any vector field on manifold [11]. Nearly Kähler manifolds satisfy

$$(\nabla_X J) X = 0,$$

for each vector field X. Then, using this definition, Blair introduced nearly cosymplectic manifold in 1971 [4] and Blair et al. defined nearly Sasakian structure in 1976 [5]. Recently, Balkan carried this notion on globally framed metric f-manifolds and he introduced and studied on nearly C manifolds [2] and nearly Kenmotsu f-manifolds [1].

The notion of globally framed manifold or globally framed f-manifold, which is generalization of complex and contact manifolds, was introduced by Nakagawa in 1966 [16]. Then, Blair defined three classes of globally framed manifolds, called K-manifold, S-manifold and C-manifold [3]. Many researchers studied on these manifolds. Falcitelli and Pastore introduced almost Kenmotsu f-manifolds in 2007 [8]. In 2014, Öztürk et al. defined almost α-cosymplectic f-manifolds, which are generalization of almost C-manifolds and almost Kenmotsu f-manifolds [18].

Tensor properties are so important in differential geometry, in particular in Riemannian geometry. Many researchers focused on many aspect of this topic. Wong studied recurrent tensor fields on a manifold endowed with a linear connection [24]. Levy proved that on a space of constant curvature, second order symmetric parallel non-singular tensors are constant multiples of the metric tensor [15]. Najafi and Hosseinpour Kashani considered this topic for nearly Kenmotsu f manifolds [17].

Now, let $(M,\ g)$ be a Riemannian manifold. If a $(0,\ 2)$-tensor field α satisfies $\nabla\alpha = \lambda \otimes \alpha$ for some 1-form λ, then it is said to be a recurrent tensor field on $(M,\ g)$. Here, the 1-form λ is called the recurrence co-vector of α. It is easy to see that every multiple of the metric tensor is a recurrent tensor. Furthermore, if α is called a closed recurrent tensor. Also we can say that the set of closed recurrent tensors contains the set of parallel tensors as a subset, for $\lambda = 0$ ([24], [25]).

In the present study, we focus on nearly Kenmotsu f-manifolds motivated by previous studies. Firstly, we prove that a second-order symmetric closed recurrent tensor is a multiple of the associated metric tensor on nearly Kenmotsu f-manifolds. Then, we get some necessary condition under which a vector field on a nearly Kenmotsu f-manifold will be a strict generalized contact or Killing vector field. Finally, we show that every φ-recurrent

nearly Kenmotsu f-manifold is an Einstein manifold of globally framed type and every locally φ-recurrent nearly Kenmotsu f-manifold is a manifold of constant curvature -1.

2. Preliminaries

Let M be $(2n+s)$-dimensional manifold and φ is a non-null $(1,\ 1)$ tensor field on M. If φ satisfies

$$\varphi^3 + \varphi = 0, \tag{2.1}$$

then φ is called an f-structure and M is called f-manifold [26]. If $rank\varphi = 2n$, namely $s = 0$, φ is called almost complex structure and if $rank\varphi = 2n+1$, namely $s = 1$, then φ reduces an almost contact structure [10]. $rank\varphi$ is always constant [21].

On an f-manifold M, P_1 and P_2 operators are defined by

$$P_1 = -\varphi^2, \quad P_2 = \varphi^2 + I, \tag{2.2}$$

which satisfy

$$\begin{aligned} P_1 + P_2 = I, \qquad & P_1^2 = P_1, \qquad P_2^2 = P_2, \\ \varphi P_1 = P_1\varphi = \varphi, \quad & P_2\varphi = \varphi P_2 = 0. \end{aligned} \tag{2.3}$$

These properties show that P_1 and P_2 are complement projection operators. There are D and $D^{\perp}$ distributions with respect to P_1 and P_2 operators, respectively [27]. Also, $\dim(D) = 2n$ and $\dim\left(D^{\perp}\right) = s$.

Let M be $(2n+s)$-dimensional f-manifold and φ is a $(1,\ 1)$ tensor field, ξ_i is vector field and η^i is 1-form for each $1 \le i \le s$ on M, respectively. If $\left(\varphi,\ \xi_i,\ \eta^i\right)$ satisfy

$$\eta^j(\xi_i) = \delta_i^j, \tag{2.4}$$

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{2.5}$$

then $\left(\varphi,\ \xi_i,\ \eta^i\right)$ is called globally framed f-structure or simply framed f-structure and M is called globally framed f-manifold or simply framed f-manifold [16]. For a framed f-manifold M, the following properties are satisfied [16]:

$$\varphi\xi_i = 0, \tag{2.6}$$

$$\eta^i \circ \varphi = 0. \tag{2.7}$$

If on a framed f-manifold M, there exists a Riemannian metric which satisfies

$$\eta^i(X) = g(X,\ \xi_i), \tag{2.8}$$

and

$$g(\varphi X,\ \varphi Y) = g(X,\ Y) - \sum_{i=1}^{s} \eta^i(X)\,\eta^i(Y), \tag{2.9}$$

for all vector fields $X,\ Y$ on M, then M is called framed metric f-manifold [9]. On a framed metric f-manifold, fundamental 2-form Φ defined by

$$\Phi(X,\ Y) = g(X,\ \varphi Y), \tag{2.10}$$

for all vector fields $X,\ Y \in \chi(M)$ [9]. For a framed metric f-manifold,

$$N_\varphi + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i, \tag{2.11}$$

is satisfied, M is called normal framed metric f-manifold, where N_φ denotes the Nijenhuis torsion tensor of φ [12].

A globally framed metric f-manifold M is called Kenmotsu f-manifold if it satisfies

$$(\nabla_X \varphi)\,Y = \sum_{k=1}^{s} \left\{ g(\varphi X,\ Y)\,\xi_k - \eta^k(Y)\,\varphi X \right\}, \tag{2.12}$$

for all vector fields $X,\ Y \in \chi(M)$ [18]. Furthermore, if a globally framed metric f-manifold M satisfies

$$(\nabla_X \varphi)\,Y + (\nabla_Y \varphi)\,X = -\sum_{k=1}^{s} \left\{ \eta^k(X)\,\varphi Y + \eta^k(Y)\,\varphi X \right\} \tag{2.13}$$

then it is called nearly Kenmotsu f-manifold. It is easily seen that every Kenmotsu f-manifold is a nearly Kenmotsu f-manifold, but the converse is not true. When a normal Kenmotsu f-manifold M is normal, it is Kenmotsu f-manifold [1]. On a nearly Kenmotsu f-manifold M, the following identities hold:

$$R(\xi_i,\ X)\,Y = \sum_{k=1}^{s} \left\{ -g(X,\ Y)\,\xi_k + \eta^k(Y)\,X \right\}, \tag{2.14}$$

$$R(X,\ Y)\,\xi_i = \sum_{k=1}^{s} \left\{ \eta^k(X)\,Y - \eta^k(Y)\,X \right\}, \tag{2.15}$$

$$S(\varphi X,\ \varphi Y) = S(X,\ Y) + (2n+s-1)\sum_{k=1}^{s} \eta^k(X)\,\eta^k(Y), \tag{2.16}$$

$$\left(\nabla_X \eta^i\right) Y = g(X,\ Y) - \sum_{k=1}^{s} \eta^k(X)\,\eta^k(Y), \tag{2.17}$$

$$\sum_{k=1}^{s} \eta^k(R(X,\ Y)\,Z) = \sum_{k=1}^{s} \left\{ g(X,\ Z)\,\eta^k(Y) - g(Y,\ Z)\,\eta^k(X) \right\}, \tag{2.18}$$

for any vector fields $X,\ Y$ on M [1].

A vector field X on a nearly Kenmotsu f-manifold M is said to be a generalized contact vector field, if

$$L_X \eta^k(Y) = \sigma \eta^k(Y) \tag{2.19}$$

or a conformal vector field, if

$$L_X g(Y,\ Z) = \rho g(Y,\ Z), \tag{2.20}$$

for any vector fields Y and Z on M, where σ and ρ are scalar function defined on M and L_X denotes the Lie derivative along X. Moreover, X is called strict generalized contact vector field or Killing vector field if $\sigma = 0$ or $\rho = 0$.

3. Recurrent Tensor Fields of the Second Order On Nearly Kenmotsu f-manifolds

Theorem 3.1. *Let M be a nearly Kenmotsu f-manifold. Then a second-order symmetric closed recurrent tensor field whose recurrence co-vector annihilates ξ_k is a multiple of the metric tensor g for each $1 \leq k \leq s$.*

Proof. We suppose that M is a nearly Kenmotsu f-manifold and α is a closed recurrent $(0,\ 2)$-tensor on M which satisfies $\lambda(\xi_k) = 0$, for each $1 \leq k \leq s$. After a straightforward calculation, we obtain

$$\alpha(R(W,\ X)Y,\ Z) + \alpha(Y,\ R(W,\ X)Z) = \lambda(W)\alpha(\nabla_X Y,\ Z) - \lambda(X)\alpha(\nabla_W Y,\ Z), \tag{3.21}$$

for any vector fields $X,\ Y,\ Z,\ W$ on M. Putting $Y = Z = W = \xi_i$ in (3.21) and using $\nabla_X \xi_i = -\varphi^2 X$, then in view of $\lambda(\xi_i) = 0$ we have

$$\alpha(R(\xi_k,\ X)\xi_k,\ \xi_k) + \alpha(\xi_k,\ R(\xi_k,\ X)\xi_k) = 0. \tag{3.22}$$

By using (2.14) and (2.15) in (3.22), we get

$$g(X,\ \xi_i)\sum_{k=1}^{s}\{\alpha(\xi_k,\ \xi_i) + \alpha(\xi_i,\ \xi_k)\} - \alpha(X,\ \xi_i) - \alpha(\xi_i,\ X) = 0 \tag{3.23}$$

Differentiating (3.23) along Y and using $\nabla_{\xi_k}\xi_k = 0$, it follows that

$$\begin{aligned}\{g(\nabla_Y X,\ \xi_i) + g(X,\ \nabla_Y \xi_i)\}&\sum_{k=1}^{s}\{\alpha(\xi_k,\ \xi_i) + \alpha(\xi_i,\ \xi_k)\} \\ = \ \alpha(\nabla_Y X,\ \xi_i) + \alpha(X,\ \nabla_Y \xi_i) &+ \alpha(\nabla_Y \xi_i,\ X) + \alpha(\xi_i,\ \nabla_Y X).\end{aligned} \tag{3.24}$$

Replacing X by $\nabla_Y X$ in (3.24), we derive

$$g(\nabla_Y X,\ \xi_i)\sum_{k=1}^{s}\{\alpha(\xi_k,\ \xi_i)+\alpha(\xi_i,\ \xi_k)\}-\alpha(\nabla_Y X,\ \xi_i)-\alpha(\xi_i,\ \nabla_Y X)=0 \tag{3.25}$$

From (3.24) and (3.25), we deduce

$$g(X,\ \nabla_Y \xi_i)\sum_{k=1}^{s}\{\alpha(\xi_k,\ \xi_i)+\alpha(\xi_i,\ \xi_k)\}=\alpha(X,\ \nabla_Y \xi_i)+\alpha(\nabla_Y \xi_i,\ X). \tag{3.26}$$

Taking in account of $\nabla_X \xi_i=-\varphi^2 X$, then we conclude that

$$g\left(X,\ Y-\sum_{k=1}^{s}\eta^k(Y)\xi_k\right)\sum_{k=1}^{s}\{\alpha(\xi_k,\ \xi_i)+\alpha(\xi_i,\ \xi_k)\} \tag{3.27}$$

$$=\ \alpha\left(X,\ Y-\sum_{k=1}^{s}\eta^k(Y)\xi_k\right)+\alpha\left(Y-\sum_{k=1}^{s}\eta^k(Y)\xi_k,\ X\right) \tag{3.28}$$

Using (3.23) and (3.27), we find

$$\alpha^{\circ}(X,\ Y)=\sum_{k=1}^{s}\alpha^{*}(\xi_k,\ \xi_i)\,g(X,\ Y). \tag{3.29}$$

Here, α° denotes the symmetric part of α defined by

$$\alpha^{\circ}(X,\ Y)=\frac{s}{2}\{\alpha(X,\ Y)+(Y,\ X)\}$$

and $\alpha^{*}(\xi_k,\ \xi_i)=\alpha(\xi_k,\ \xi_i)+\alpha(\xi_i,\ \xi_k)$. Furthermore, by using (3.23) and $\nabla\alpha=\lambda\otimes\alpha$, then we have $\nabla_X\mu=\lambda(X)\mu$, where X is an arbitrary vector field on M and

$$\mu=\sum_{k=1}^{s}\alpha^{*}(\xi_k,\ \xi_i).$$

Hence, if α is a parallel tensor or equivalently $\lambda=0$, so we can say μ is a constant function, but in general μ is not a constant function. Additionally, if α is symmetric, i.e.$\alpha=\alpha^{\circ}$, then we conclude $\alpha=\mu g$ and $\lambda=d\mu$.

4. Geometric Vector Fields on Nearly Kenmotsu f-manifolds

Theorem 4.1. *Every generalized contact vector field on a nearly Kenmotsu f-manifold leaving the Ricci tensor invariant is a generalized strict contact vector field.*

Proof. Let us suppose that a generalized contact vector field X leaves the Ricci tensor invariant, i.e.

$$L_X S(Y,\ Z)=0, \tag{4.30}$$

for any vector fields Y and Z on M. Taking $Y=\xi_i$ in (4.30), it implies that

$$L_X(S(Y,\ \xi_i))=S(L_X Y,\ \xi_i)+S(Y,\ L_X\xi_i). \tag{4.31}$$

By using (2.16), (2.19) and (4.31), then we have

$$(1-(2n+s))\,\sigma\sum_{k=1}^{s}\eta^{k}(Y)=S(Y,\ L_X\xi_i)\,. \tag{4.32}$$

Taking $Y=\xi_j$ in (4.32) and using (2.16), then we obtain

$$\sigma=\sum_{k=1}^{s}\eta^{k}(L_X\xi_i)\,. \tag{4.33}$$

On the other hand, substituting ξ_i for Y in (2.19) it follows that

$$\sigma=-\sum_{k=1}^{s}\eta^{k}(L_X\xi_i)\,, \tag{4.34}$$

which means $\sigma=0$.

Theorem 4.2. *Every vector field on a nearly Kenmotsu f-manifold leaving the curvature tensor invariant is a Killing vector field.*

Proof. For a vector field X on a nearly Kenmotsu f-manifold, we assume that $L_XR=0$. It is well-known that the curvature tensor of g satisfies

$$g(R(U,\ V)Y,\ Z)+g(R(U,\ V)Z,\ Y)=0, \tag{4.35}$$

for all vector fields $U,\ V,\ Y,\ Z$ on M. Applying L_X to (4.35), we have

$$L_Xg(R(U,\ V)Y,\ Z)+L_Xg(R(U,\ V)Z,\ Y)=0. \tag{4.36}$$

Setting $U=Y=Z=\xi_i$ in (4.36) and using (2.14), we derive

$$L_Xg(V,\ \xi_i)=\eta^{i}(V)\,L_Xg(\xi_i,\ \xi_i)\,. \tag{4.37}$$

On the other hand, putting $U=Y=\xi_i$ in (4.36) and using (2.14), it implies that

$$\begin{aligned} 0 \;=\;& L_Xg(V,\ Z)-\eta^{i}(V)\sum_{k=1}^{s}L_Xg(\xi_k,\ Z) \\ &+L_Xg(\xi_i,\ V)\sum_{k=1}^{s}\eta^{k}(Z)-g(V,\ Z)\,L_Xg(\xi_i,\ \xi_i) \end{aligned} \tag{4.38}$$

From (4.37) and (4.38), then we get

$$L_Xg(V,\ Z)=\rho g(V,\ Z)\,, \tag{4.39}$$

where $\rho=g(\xi_i,\ \xi_i)$. Under the assumption $L_XR=0$, we see that $L_XS=0$. Furthermore, it is said to be

$$\rho=-2g(L_X\xi_i,\ \xi_i)=\frac{2}{2n+s-1}S(L_X\xi_i,\ \xi_i)=\frac{1}{(1-2n-s)}L_XS(\xi_i,\ \xi_i)=0. \tag{4.40}$$

5. φ-Recurrent Nearly Kenmotsu f-manifolds

Firstly, we give some basic definitions.

Definition 5.1. *A nearly Kenmotsu f-manifold M is said to be locally φ-symmetric manifold in the sense of Takahashi* [22] *if it satisfies*

$$\varphi^2 \left(\left(\nabla_W R \right) (X,\ Y) Z \right) = 0, \tag{5.41}$$

for all vector fields $X,\ Y,\ Z,\ W$ orthogonal to ξ_k, for each $1 \leq k \leq s$.

Definition 5.2. *A nearly Kenmotsu f-manifold M is said to be φ-recurrent manifold in the sense of Takahashi* [22] *(locally φ-recurrent manifold, resp.) if there exists a nonzero 1-form B such that*

$$\varphi^2 \left(\left(\nabla_W R \right) (X,\ Y) Z \right) = B(W) R(X,\ Y) Z, \tag{5.42}$$

for arbitrary vector fields $X,\ Y,\ Z,\ W$ (for all $X,\ Y,\ Z,\ W$ orthogonal to ξ_k, for each $1 \leq k \leq s$).

Theorem 5.1. *Let M be an η-Einstein nearly Kenmotsu f-manifold. If at least one of the coefficients is constant function, then M is an Einstein manifold.*

Proof. From (5.42), we have

$$\left(\nabla_W R \right) (X,\ Y) Z = \sum_{k=1}^{s} \eta^k \left(\left(\nabla_W R \right) (X,\ Y) Z \right) \xi_k - B(W) R(X,\ Y) Z. \tag{5.43}$$

By using (5.43) and Bianchi identity, we obtain

$$B(W) \sum_{k=1}^{s} \eta^k \left(R(X,\ Y) Z \right) + B(X) \sum_{k=1}^{s} \eta^k \left(R(Y,\ W) Z \right) + B(Y) \sum_{k=1}^{s} \eta^k \left(R(W,\ X) Z \right) = 0. \tag{5.44}$$

Now, let $\{e_i\},\ 1 \leq i \leq 2n+s$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $Y = Z = e_i$ in (5.44) and taking summation over i, in view of (2.14) and (2.15), then we conclude that

$$B(W) \sum_{k=1}^{s} \eta^k (X) = B(X) \sum_{k=1}^{s} \eta^k (W), \tag{5.45}$$

for any vector fields $X,\ W$. Replacing X by ξ_i in (5.45), it implies that

$$B(W) = \eta^i \left(\widehat{B} \right) \sum_{k=1}^{s} \eta^k (W), \tag{5.46}$$

where $B(\xi_i) = g\left(\xi_i,\ \widehat{B}\right) = \eta^i\left(\widehat{B}\right)$. Now, let us suppose that M is η-Einstein, the we can write

$$S(X,\ Y) = ag(X,\ Y) + b\sum_{k=1}^{s}\eta^k(X)\,\eta^k(Y), \tag{5.47}$$

where a and b are scalar functions on M. Taking $Y = \xi_i$ in (2.17), from (5.47) we deduce

$$a + b = 1 - 2n - s. \tag{5.48}$$

Using local coordinate, we can rewrite (5.47) as follows:

$$R_{ij} = ag_{ij} + b\sum_{k=1}^{s}\eta_i^k\eta_j^k, \tag{5.49}$$

which implies

$$r = (2n + s)\,a + sb. \tag{5.50}$$

Taking the covariant derivative with respect to g from (5.49), we derive

$$R_{ij,m} = a_{,m}g_{ij} + \sum_{k=1}^{s}\left\{b_{,m}\eta_i^k\eta_j^k + b\eta_{i,m}^k\eta_j^k + b\eta_i^k\eta_{j,m}^k\right\}. \tag{5.51}$$

By contracting (5.51) with g^{im}, we get

$$R_{j,m}^m = a_{,j} + \sum_{k=1}^{s}\left\{b_{,m}\xi^m\eta_j^k + b\eta_{i,m}^k g^{im}\eta_j^k + b\eta_i^k\eta_{j,m}^k g^{im}\right\}. \tag{5.52}$$

We know that $R_{j,m}^m = \dfrac{1}{2}r_{,j}$. Thus we have

$$r_{,j} = 2\left\{a_{,j} + \sum_{k=1}^{s}\left[b_{,m}\xi^m + 2nb\right]\eta_j^k\right\}. \tag{5.53}$$

Here, we use (2.17) and $\eta_{i,m}g^{im} = \left\{g_{im} - \sum_{k=1}^{s}\eta_i^k\eta_m^k\right\}g^{im} = 2n$. Moreover, taking the covariant derivative of (5.48) and from (5.50), then we obtain

$$r_{,j} = 2na_{,j}. \tag{5.54}$$

Substituting (5.54) into (5.53), it follows that

$$na_{,j} = a_{,j} + \sum_{k=1}^{s}\left[b_{,m}\xi^m + 2nb\right]\eta_j^k. \tag{5.55}$$

By contracting (5.55) with ξ^j and using (5.48), we deduce

$$b_{,m}\xi^m = -2b. \tag{5.56}$$

Moreover, if b or a is a constant function, then (5.56) implies that $b = 0$. Hence, M is an Einstein manifold.

Theorem 5.2. *Every φ-recurrent nearly Kenmotsu f-manifold is an Einstein manifold.*

Proof. By using (5.43), we obtain

$$-g\left(\left(\nabla_W R\right)(X,\ Y)Z,\ U\right)+\sum_{k=1}^{s}\eta^k\left(\left(\nabla_W R\right)(X,\ Y)Z\right)\eta^k(U)=B(W)\,g\left(R(X,\ Y)Z,\ U\right). \tag{5.57}$$

Let $\{e_i\}$, $1\leq i\leq 2n+s$ be an orthonormal basis of the tangent space at any point of the manifold M. Setting $X=U=e_i$ in (5.57) and taking summation over i, then we deduce that

$$-\left(\nabla_W S\right)(Y,\ Z)+\sum_{i=1}^{2n+s}\eta^i\left(\left(\nabla_W R\right)(e_i,\ Y)Z\right)\eta^i(e_i)=B(W)\,S(Y,\ Z). \tag{5.58}$$

Replacing Z by ξ_k in (5.58), we have

$$-\left(\nabla_W S\right)(Y,\ \xi_k)+\sum_{i=1}^{2n+s}\eta^i\left(\left(\nabla_W R\right)(e_i,\ Y)\xi_k\right)\eta^i(e_i)=B(W)\,S(Y,\ \xi_k). \tag{5.59}$$

Now, we will show that $\sum_{i=1}^{2n+s}\eta^i\left(\left(\nabla_W R\right)(e_i,\ Y)\xi_k\right)\eta^i(e_i)$ vanishes identically. Firstly, we recall

$$\begin{aligned}\sum_{i=1}^{2n+s}\eta^i\left(\left(\nabla_W R\right)(e_i,\ Y)\xi_k\right)\eta^i(e_i) &= \sum_{k=1}^{s}\eta^k\left(\left(\nabla_W R\right)(e_i,\ Y)\xi_k\right) \\ &= \sum_{k=1}^{s}g\left(\left(\nabla_W R\right)(e_k,\ Y)\xi_k,\ \xi_k\right),\end{aligned} \tag{5.60}$$

where we use $\eta^i(e_i)=0$ for $i=1,\ \ldots,2n$. From the properties, we find

$$\begin{aligned}&\sum_{k=1}^{s}g\left(\left(\nabla_W R\right)(e_k,\ Y)\xi_k,\ \xi_k\right) \\ =\ &\sum_{k=1}^{s}\left\{g\left(\nabla_W R(e_k,\ Y)\xi_k,\ \xi_k\right)-g\left(R(\nabla_W e_k,\ Y)\xi_k,\ \xi_k\right)\right. \\ &\left.-g\left(R(e_k,\ \nabla_W Y)\xi_k,\ \xi_k\right)-g\left(R(e_k,\ Y)\nabla_W\xi_k,\ \xi_k\right)\right\}.\end{aligned} \tag{5.61}$$

Making use of (5.61) at $p\in M$ and using $g_{ij}(p)=\delta_{ij}$, we conclude that $\nabla_W e_k(p)=0$. On the other hand, we get

$$\sum_{k=1}^{s}g\left(R(e_k,\ \nabla_W Y)\xi_k,\ \xi_k\right)=-\sum_{k=1}^{s}g\left(R(\xi_k,\ \xi_k)\nabla_W Y,\ e_k\right)=0, \tag{5.62}$$

since R skew-symmetric. By virtue of (5.62) and $\nabla_W e_k(p)=0$ in (5.61), we derive

$$\begin{aligned}\sum_{k=1}^{s}g\left(\left(\nabla_W R\right)(e_k,\ Y)\xi_k,\ \xi_k\right) &= \sum_{k=1}^{s}\left\{g\left(\nabla_W R(e_k,\ Y)\xi_k,\ \xi_k\right)\right. \\ &\quad\left.-g\left(R(e_k,\ Y)\nabla_W\xi_k,\ \xi_k\right)\right\}.\end{aligned} \tag{5.63}$$

By using $g(R(e_k,\ Y)\xi_k,\ \xi_k) = -g(R(\xi_k,\ \xi_k)Y,\ e_k) = 0$, we find

$$\sum_{k=1}^{s} \{g(\nabla_W R(e_k,\ Y)\xi_k,\ \xi_k) - g(R(e_k,\ Y)\xi_k,\ \nabla_W \xi_k)\} = 0, \tag{5.64}$$

which implies

$$\begin{aligned} 0 &= \sum_{k=1}^{s} g((\nabla_W R)(e_k,\ Y)\xi_k,\ \xi_k) \qquad (5.65) \\ &= -\sum_{k=1}^{s} \{g(R(e_k,\ Y)\xi_k,\ \nabla_W \xi_k) + g(R(e_k,\ Y)\nabla_W \xi_k,\ \xi_k)\}, \end{aligned}$$

since R skew-symmetric. Hence, we prove $\sum_{i=1}^{2n+s} \eta^i((\nabla_W R)(e_i,\ Y)\xi_k)\eta^i(e_i) = 0$ and from (5.59) we have

$$-(\nabla_W S)(Y,\ \xi_k) = B(W)S(Y,\ \xi_k). \tag{5.66}$$

Furthermore, it is well-known that

$$(\nabla_W S)(Y,\ \xi_k) = \nabla_W S(Y,\ \xi_k) - S(\nabla_W Y,\ \xi_k) - S(Y,\ \nabla_W \xi_k). \tag{5.67}$$

By applying (2.16), (2.17) and $\nabla_X \xi_i = -\varphi^2 X$ in (5.67), it follows

$$(\nabla_W S)(Y,\ \xi_k) = -(2n+s-1)g(Y,\ W) - S(Y,\ W). \tag{5.68}$$

Plugging (5.68) into (5.66) and using (5.46), we conclude that

$$S(Y,\ W) = (1-2n-s)g(Y,\ W) + (1-2n-s)\eta^i\left(\widehat{B}\right)\sum_{k=1}^{s} \eta^k(Y)(W),$$

which means the manifold η-Einstein of globally framed type with $a = (1-2n-s)$ is constant. By Theorem 4., it is said to be M is an Einstein manifold

Theorem 5.3. *A locally φ-recurrent nearly Kenmotsu f-manifold has constant curvature -1.*

Proof. Differentiating (2.15) with respect to any vector field W and taking in account of (2.17), after an easy calculation we find

$$(\nabla_W R)(X,\ Y)\xi_i = g(W,\ X)Y - g(W,\ Y)X - R(X,\ Y)W. \tag{5.69}$$

By using (2.18) and from (5.69), we get

$$\sum_{k=1}^{s} \eta^k((\nabla_W R)(X,\ Y)\xi_k) = 0. \tag{5.70}$$

From (5.69) and (5.70), we have from (5.43)

$$\sum_{k=1}^{s}\left(\nabla_{W}R\right)(X,\ Y)\,\xi_k = B(W)\sum_{k=1}^{s}R(X,\ Y)\,\xi_k. \tag{5.71}$$

By virtue of (5.69), it implies that

$$-\,g(W,\ X)\,Y + g(W,\ Y)\,X + R(X,\ Y)\,W = B(W)\sum_{k=1}^{s}R(X,\ Y)\,\xi_k. \tag{5.72}$$

Thus, if X and Y are orthogonal to ξ_k for each $1 \leq k \leq s$, we derive

$$\sum_{k=1}^{s}R(X,\ Y)\,\xi_k = 0. \tag{5.73}$$

Hence, for all vector fields $X,\ Y$ and W, we deduce

$$R(X,\ Y)\,W = -\left\{g(W,\ X)\,Y + g(W,\ Y)\,X\right\},$$

which gives us desired result.

6. Example

Let M be a 6-dimensional manifold given by

$$M = \left\{(x_1,\ x_2,\ y_1,\ y_2,\ z_1,\ z_2) \in \mathbb{R}^6 :\ z_1,\ z_2 \neq 0\right\}$$

where $(x_1,\ x_2,\ y_1,\ y_2,\ z_1,\ z_2)$ are standard coordinates in $\mathbb{R}^6$. We choose the vector fields as in the following:

$$\begin{aligned} e_1 &= e^{-(z_1+z_2)}\frac{\partial}{\partial x_1}, & e_2 &= e^{-(z_1+z_2)}\frac{\partial}{\partial x_2},\\ e_3 &= e^{-(z_1+z_2)}\frac{\partial}{\partial y_1}, & e_4 &= e^{-(z_1+z_2)}\frac{\partial}{\partial y_2},\\ e_5 &= \frac{\partial}{\partial z_1}, & e_6 &= \frac{\partial}{\partial z_2}. \end{aligned}$$

which are linearly independent at any point of M. Denote g the Riemannian metric defined by

$$g = e^{2(z_1+z_2)}\sum_{i=1}^{2}\left\{dx_i \otimes dx_i + dy_i \otimes dy_i + dz_i \otimes dz_i\right\}.$$

Let η_1 and η_2 be 1-forms given by $\eta_1(X) = g(X,\ e_5)$ and $\eta_2(X) = g(X,\ e_6)$ for any vector field on M, respectively. Thus $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6\}$ is an orthornormal basis of tangent space at any point on M. We define the $(1,1)$-tensor field φ as follows:

$$\varphi\left(\sum_{i=1}^{2}\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i} + z_i\frac{\partial}{\partial z_i}\right)\right) = \sum_{i=1}^{2}\left(x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i}\right).$$

Hence we derive

$$\varphi e_1 = e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2, \quad \varphi e_5 = 0, \quad \varphi e_6 = 0.$$

By virtue of the linearity of g and φ, we deduce that

$$\eta_1(e_5) = 1, \quad \eta_2(e_6) = 1, \quad \varphi^2 X = -X + \eta_1(X)\, e_5 + \eta_2(X)\, e_6$$
$$g(\varphi X,\ \varphi Y) = g(X,\ Y) - \eta_1(X)\,\eta_1(Y) - \eta_2(X)\,\eta_2(Y).$$

Then for $\xi_1 = e_5$ and $\xi_2 = e_6$, $\left(\varphi,\ \xi_i,\ \eta^i,\ g\right)$ defines a globally framed metric f-structure on M. It is clear that the 1-forms are closed. On the other hand, we get

$$\Phi\left(\frac{\partial}{\partial x_i},\ \frac{\partial}{\partial y_i}\right) = g\left(\frac{\partial}{\partial x_i},\ \varphi\frac{\partial}{\partial y_i}\right) = g\left(\frac{\partial}{\partial x_i},\ -\frac{\partial}{\partial x_i}\right) = e^{-2(z_1+z_2)}$$

which means that $\Phi = -e^{2(z_1+z_2)}$. Therefore, we obtain

$$d\Phi = -2e^{2(z_1+z_2)}\,(dz_1 + dz_2) \wedge dx \wedge dy = 2\,(\eta_1 + \eta_2) \wedge \Phi$$

which gives us M is an almost Kenmotsu f-manifold. After some easy computations, it is clearly seen that the Nijenhuis tensor field vanishes identically, that is, M is normal. So M is a Kenmotsu f-manifold. It is well-known that every Kenmotsu f-manifold is a nearly Kenmotsu f-manifold (see [2]). Thus we conclude that M is a nearly Kenmotsu f-manifold

Furthermore we have

$$[e_1,\ e_5] = [e_1,\ e_6] = e_1,$$
$$[e_2,\ e_5] = [e_2,\ e_6] = e_2,$$
$$[e_3,\ e_5] = [e_3,\ e_6] = e_3,$$
$$[e_4,\ e_5] = [e_4,\ e_6] = e_3$$

and remaning terms $[e_i,\ e_j] = 0$ for all $1 \leq i,\ j \leq 6$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is defined by

$$\begin{aligned} 2g(\nabla_X Y,\ Z) &= Xg(Y,\ Z) + Yg(Z,\ X) - Zg(X,\ Y) \\ &\quad -g(X,\ [Y,\ Z]) - g(Y,\ [X,\ Z]) + g(Z,\ [X,\ Y]). \end{aligned}$$

By using this Koszul's formula, then we obtain

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = -(e_5 + e_6)$$

and the other terms $\nabla_{e_i} e_j = 0$ for all $1 \leq i,\ j \leq 6$. It is welknown that Riemannian curvature tensor is defined by

$$R(X,\ Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,\ Y]} Z \tag{6.74}$$

for any vector fields on M. By the above results, we can easily get the non-vanishing components of the Riemannian curvature tensors as in the following:

$$\begin{aligned} R(e_1,\ e_5)e_1 = R(e_1,\ e_6)e_1 = e_5 + e_6, \\ R(e_2,\ e_5)e_2 = R(e_2,\ e_6)e_2 = e_5 + e_6, \\ R(e_3,\ e_5)e_3 = R(e_3,\ e_6)e_3 = e_5 + e_6, \\ R(e_4,\ e_5)e_4 = R(e_4,\ e_6)e_4 = e_5 + e_6. \end{aligned} \tag{6.75}$$

Now, let $X,\ Y$ and Z be three vector fields given by

$$\begin{aligned} X &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6, \\ Y &= b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6, \\ Z &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 \end{aligned}$$

where $a_i,\ b_i$ and c_i are all non-zero real numbers for all $i = 1,\ \ldots,\ 6$. By taking into account of (6.75) in (6.74), then we get

$$R(X,\ Y)Z = \{a_1c_1 + a_2c_2 + a_3c_3 + a_4c_4\}(b_5 + b_6)(e_5 + e_6).$$

Again by using (6.75), then we obtain the scalar curvature $r = 8$. By these considerations, it is said that the 6-dimensional manifold M satisfies Theorem 2 and Theorem 3.

7. Conclusion

In this paper, we study some tensor conditions on nearly Kenmotsu f-manifold and we generalize some previous results obtain by Najafi and Hosseinpour in [17] since a nearly Kenmotsu f-manifold is a nice generalization of nearly Kenmotsu one. Additonally, we construct an example satisfying some corresponding results.

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ON SEMI-INVARIANT $\xi^{\perp}$-SUBMANIFOLDS OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. In the present paper, we study semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds. We discuss the integrability conditions of the distributions D and $D^{\perp}$ on semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds. We also obtain some characterizations for the totally umbilical semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds.

1. Introduction

In 1989, K. Motsumoto [1] introduced the notion of Lorentzian para-Sasakian manifold (LP-Sasakian manifold). I. Mihai and R. Rosca [2] defined the same notion independently and thereafter many authors [3, 4, 5] studied LP-Sasakian manifolds. M.M. Tripathi and U.C. De [6] studied submanifolds of a Lorentzian almost paracontact manifold. C. Ozgur [7] studied invariant submanifolds of LP Sasakian manifolds. In 1981, A. Bejancu [8] introduced the notion of semi-invariant submanifold or contact CR-submanifold, as a generalization of invariant and anti-invariant submanifolds of an almost contact metric manifold. P. Alegre [9] studied semi-invariant submanifolds of Lorentzian para-Sasakian manifold. CR-submanifolds

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of LP-Saskian manifold were studied by several geometers (see, [10], [11], [12], [13], [14]). N. Papaghuic [15] defined $\xi^{\perp}$-submanifolds in which the structural vector field ξ is orthogonal to the submanifolds and studied geometry of the leaves on Kenmotsu manifold. Constantin C. et. al [16] studied semi-invariant $\xi^{\perp}$-submanifolds of generalized quasi-Sasakian manifolds. M. M. Tripathi [17] studied semi-invariant $\xi^{\perp}$-submanifolds of trans-Sasakian manifold. Further, S.Y. Perktas et. al [18] studied semi-invariant $\xi^{\perp}$-submanifolds of P-Sasakian manifold. In this paper, we study semi-invariant $\xi^{\perp}$-submanifolds of LP-Sasakian manifold. In particular, we recover the results of Papaghiuc [15] and Calin [16].

The paper is organized as follows. In section 2, we give a breif description of Lorentzian para-Sasakian manifold. In section 3, we find some results on semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds, discuss the integrability of distributions D and $D^{\perp}$ of semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds and finally in section 4, we find some characterizations for the totally umbilical semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds.

2. **Preliminaries**

Lorentzian para-Sasakian manifold

Let $\bar{M}$ be $(2n+1)-$dimensional almost contact metric manifold with a metric tensor g, a tensor field ϕ of type $(1,1)$, a vector field ξ and a $1-$form η which satisfy

$$\phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, Y) = g(X, \phi Y) \tag{2.4}$$

for all vector fields X, Y tangent to $\bar{M}$. Such a manifold is termed as Lorentzian para-contact manifold and the structure (ϕ, η, ξ, g) a Lorentzian para-contact structure [1]. Also in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi = 0,\ \eta(\phi X) = 0, \mathrm{rank}(\phi) = n - 1.$$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian (LP-Sasakian manifold if [2]).

$$(\bar{\nabla}_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

$$\bar{\nabla}_X \xi = \phi X \tag{2.6}$$

for all vector fields X, Y tangent to $\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection with respect to g.

3. Semi-invariant $\xi^{\perp}$-submanifolds

Let M be an m-dimensional submanifold of $\bar{M}$, isometrically immersed in $\bar{M}$. The tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as

$$T\bar{M} = TM \oplus TM^{\perp}.$$

Definition 3.1 [8] An m-dimensional Riemannian submanifold M of a Lorentzian para-Sasakian manifold $\bar{M}$ is called a semi-invariant $\xi^{\perp}$-submanifold of Lorentzian para-Sasakian manifold if ξ is normal to M and there exists on M a pair of distributions $(D, D^{\perp})$ such that
(i) TM orthogonally decomposes as $D \oplus D^{\perp}$,
(ii) the distribution D_x is invariant under ϕ, that is $\phi D_x \subset D_x$ for each $x \in M$,
(iii) the distribution $D^{\perp}$ is anti-invariant under ϕ, that is $\phi D_x^{\perp}(M) \subset T_x^{\perp}(M)$ where $T_x M$ and $T_x^{\perp} M$ are tangent and normal spaces of M at $x \in M$. If $D^{\perp} = 0$ then M is an invariant $\xi^{\perp}$-submanifold. The normal bundle $T^{\perp}M$ can also be decomposed as

$$T^{\perp}M = \phi D^{\perp} \oplus \mu \oplus \{\xi\},$$

where $\phi\mu \subseteq \mu$.

Any vector X tangent to M is given by

$$X = PX + QX, \tag{3.1}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. Moreover, for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, we put

$$\phi X = tX + \omega X, \tag{3.2}$$

where tX (resp. ωX) denotes the tangential (resp. normal) components of ϕX and

$$\phi N = BN + CN, \tag{3.3}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

Gauss formula for semi-invariant $\xi^{\perp}$-submanifolds of an $LP-$ Sasakian manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y). \tag{3.4}$$

Weingarten formula is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{3.5}$$

for any $X, Y \in TM$, $N \in T^{\perp}M$, where h(resp. A_N) is the second fundamental form (resp. tensor) of M in $\bar{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X, Y). \tag{3.6}$$

Now, we study the integrability of both the distributions D and $D^{\perp}$. For this purpose, first we establish some results for further use.

Proposition 3.1. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then*

$$(a)\ (\nabla_X t)Y = A_{\omega Y} X + Bh(X,Y), \tag{3.7}$$

$$(b)\ (\nabla_X \omega)Y = Ch(X,Y) - h(X,tY) + g(X,Y)\xi$$

$\forall\ X, Y \in \Gamma(TM)$.

Proof In view of (3.2), (3.3), (3.4) *and* (3.5)*, we have*

$$(\bar{\nabla}_X \phi)Y = (\nabla_X t)Y - A_{\omega Y} X + (\nabla_X \omega)Y + h(X,tY) - \phi h(X,Y). \tag{3.8}$$

Using (2.6) *in* (3.8)*, we get*

$$g(X,Y)\xi + \phi h(X,Y) = (\nabla_X t)Y - A_{\omega Y} X + (\nabla_X \omega)Y + h(X,tY). \tag{3.9}$$

Comparing tangential and normal components of (3.9)*, we have our assertion.*

We can state the following proposition.

Proposition 3.2 (16)**.** *Let M be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then*

(a) $BN \in D^{\perp}$,

(b) $CN \in \mu$

for any $N \in \Gamma(TM^{\perp})$.

Proposition 3.3. *Let* M *be a semi-invariant* $\xi^{\perp}$*-submanifold of an LP-Sasakian manifold* $\bar{M}$. *Then*

$$A_{\omega Z}W = A_{\omega W}Z.$$

Proof Let $Y, Z \in D^{\perp}$. *Using (2.5), (3.2), (3.4) and (3.6), we have*

$$g(A_{\phi W}Z, X) = g(h(X,Z), \phi W)$$

$$= g(\bar{\nabla}_X Z, \phi W)$$

$$= g(\phi\bar{\nabla}_X Z, W)$$

$$= g(\bar{\nabla}_X \phi Z, W)$$

$$= -g(\phi Z, \bar{\nabla}_X W)$$

$$= -g(h(X,W), \phi Z)$$

$$= -g(A_{\phi Z}W, X),$$

which is equivalent to

$$A_{\phi W}Z = A_{\phi Z}W.$$

But from (3.2), we have $\phi Z = \omega Z$ *and* $\phi W = \omega W$, *then above equation reduces to* $A_{\omega W}Z = A_{\omega Z}W$.

Theorem 3.1. *Let* M *be a semi-invariant* $\xi^{\perp}$*-submanifold of an LP-Sasakian manifold* $\bar{M}$. *Then the distribution* D *is integrable if and only if*

$$h(X, \phi Y) = h(Y, \phi X) \tag{3.10}$$

$\forall\ X, Y \in \Gamma(D)$.

Proof Let $X, Y \in \Gamma(D)$. *Then from* (3.7)(*b*), *we get*

$$\omega[X,Y] = h(X,tY) - h(Y,tX). \tag{3.11}$$

Our assertion is a consequence of (3.11).

Theorem 3.2. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then the distribution $D^{\perp}$ is integrable.*

Proof In view of (3.7)(a) and Proposition 3.3, letting $Z, W \in \Gamma(D^{\perp})$, we have

$$t[Z, W] = A_{\omega Z}W - A_{\omega W}Z = 0.$$

Consequently, $[Z, W] \in \Gamma(D^{\perp})$ for all $Z, W \in \Gamma(D^{\perp})$. Hence $D^{\perp}$ is integrable.

Suppose that $(e_i, \phi e_i, e_{2p+j}), i \in 1, 2, ..., p,\ j \in 1, 2, ..., q$ be an adapted orthonormal local frame on M, where $q = dim\ D^{\perp}$. Now, we can state the following:

Theorem 3.3. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then*

$$\eta(H) = 1/m\ trace(A_{\xi}),\ m = 2p + q.$$

Proof From the general mean curvature formula $H = 1/m \sum_{a=1}^{s} trace(A_{\xi_a})\xi_a$, where $\{\xi_1, \xi_2, ..., \xi_s\}$ is an orthonormal basis in $TM^{\perp}$, the conclusion holds by straight forward computations.

Theorem 3.4. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then*

(1) *if the distribution D is integrable, then its leaves are totally geodesic in M if and only if $h(X, Y) \in \Gamma(\mu)$, where $X, Y \in \Gamma(D)$,*

(2) *any leaf of the distribution $D^{\perp}$ is totally geodesic in M if and only if $h(X, Z) \in \Gamma(\mu)$, where $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.*

Proof Let us prove the first statement. Let M^* be a leaf of the integrable distribution D and h^* the second fundamental form of M^* in M. Also, let $X, Y \in M^*$, then $X, Y \in D$. Differentiating covariantly $\phi Y = tY$ and using (3.4), we get

$$\bar{\nabla}_X tY + h^*(X, tY) = (\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y).$$

Using (2.5) in above equation, we have

$$(\bar{\nabla}_X tY) + h^*(X, tY) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi + \phi(\bar{\nabla}_X Y).$$

Taking inner product with Z and noting that $Z\epsilon D^{\perp}$, $\phi Z\epsilon\phi D^{\perp}\subset TM^{\perp}$, $g(\phi X,Y)=g(X,\phi Y)$, we get

$$g(h^*(X,tY),Z)=g(\phi\overline{\nabla}_XY,Z)$$

$$g(h^*(X,tY),Z)=g(\overline{\nabla}_XY,\phi Z)$$

$$g(h^*(X,tY),Z)=g(\nabla_XY+h(X,Y),\phi Z)$$

$$g(h^*(X,tY),Z)=g(\nabla_XY,\phi Z)+g((h(X,Y),\phi Z)$$

$$g(h^*(X,tY),Z)=g(h(X,Y),\phi Z),$$

which gives

$$h^*(X,tY)=0,$$

if and only if $h(X,Y)\in\mu$.

The proof of second part of the theorem is analogous to that of Kenmotsu case in ([15], P. 117).

4. **Totally umbilical semi-invariant $\xi^{\perp}$-submanifolds**

In this section, we obtain a complete characterization of a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. For a totally umbilical submanifold we have

$$h(X,Y)=g(X,Y)H,\ \ X,Y\in\Gamma(TM). \tag{4.1}$$

Theorem 4.1. *A semi-invariant $\xi^{\perp}$-submanifold M of an LP-Sasakian manifold $\bar{M}$ with dim $D^{\perp}\geq 2$ is totally umbilical if and only if*

$$h(X,Y)=1/m\ g(X,Y)\ trace\ (A_{\xi})\xi. \tag{4.2}$$

Proof Suppose that M is a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Let $X\in\Gamma(D)$ be the unit vector field and $N\in\Gamma(\mu)$. Using Gauss formula (3.4)*, we get*

$$h(X,X)=-\nabla_XX+\phi(\bar{\nabla}_X\phi X-(\bar{\nabla}_X\phi)X)-\eta(\bar{\nabla}_XX)\xi.$$

$$=-\nabla_XX+\phi(\bar{\nabla}_X\phi X-(\bar{\nabla}_X\phi)X)-g(\nabla_XX+h(X,X),\xi)\xi.$$

$$=-\nabla_XX+\phi(\bar{\nabla}_X\phi X-(\bar{\nabla}_X\phi)X)$$

Taking inner product with N, we have

$$g(H,N)=g(h(X,X),N)=0,$$

which shows that $H \in \phi D^{\perp} \oplus$ span $\{\xi\}$.

Now, letting $W, Z \in D^{\perp}$, From (2.5) and (3.5), we get

$$g(W,Z)\xi + \phi(\nabla_W Z + \phi h(W,Z) = -A_{\phi Z}W + \nabla^{\perp}_W \phi Z.$$

Equating vertical components of above equations and then the inner product with ϕH gives

$$g(W,Z)g(\phi H, \phi H) = g(Z, \phi H)g(W, \phi H). \tag{4.3}$$

Since $D^{\perp} \geq 2$, for $Z = W \perp \phi H$, the above relation gives $\phi H = 0$ which implies that $H \in$ span$\{\xi\}$. If we consider an orthonormal frame $\{e_i, e_{p+i}\}, i = 1, 2, 3, ..., p$ on M. Since M is a semi-invariant $\xi^{\perp}$-submanifold, we can write

$$H = g(H,\xi)\xi = 1/m \sum g(h(e_i, e_i), \xi)\xi = 1/m \ trace(A_{\xi})\xi.$$

Using (4.1) *in above equation, we get* (4.2).

Conversely, if (4.2) *holds, then we get* (4.3). *From* (4.2) *and* (4.3) *together we conclude that M is totally umbilical.*

Corollary 4.1. *Every semi-invariant $\xi^{\perp}$-hypersurface M of an LP-Sasakian manifold is geodesic.*

Proof Let M is a hypersurface, that is $TM^{\perp} =$ span $\{\xi\}$, which implies that $h(X,Y) \in$ span ξ. Then Corollary 4.2 *follows from* (4.3).

We call a semi-invariant product as a semi-invariant $\xi^{\perp}$-submanifold of $\bar{M}$ which can be locally written as a Riemannian product of a ϕ-invariant submanifold and a ϕ anti–invariant submanifold of $\bar{M}$, both of them orthogonal to ξ.

Theorem 4.2. *Let M be a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$ with dim $D^{\perp} \geq 2$. Then M is a semi-invariant product.*

Proof Let M be a totally umbilical submanifold, then $h(X,Z) = 0$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. So by Theorem 3.4 , *the leaves of $D^{\perp}$ are totally geodesic submanifold of M. By*

Corollary 4.1*, $h(X,Y) \in span\ \{\xi\} \subset \mu$ for any $X, Y \in \Gamma(D)$. Combining this fact with Theorem* 3.4*, this implies that the invariant distribution D is integrable and its integral manifolds are totally geodesic submanifolds of M. Hence we conclude that M is semi-invariant product.*

Theorem 4.3. *Let M be a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. If D is integrable, then each leaf of D is a totally geodesic submanifold of M.*

Proof Using (3.7)(*b*) *for any $X \in \Gamma(D)$, we get*

$$\omega(\nabla_X X) = -g(X,X)CH + g(X,\phi X)H - g(X,X)\xi.$$

Since $CH \in \Gamma(\mu)$ by Proposition 3.2*, $H \in span\ \{\xi\}$ from Theorem* 4.1*, $\xi \in \Gamma(\mu)$ and $\omega(\nabla_X X) \in \phi D^{\perp}$. From the above equation we deduce that $\omega(\nabla_X X) = 0$, or equivalently*

$$\nabla_X X \in \Gamma(D)\ \forall\ X \in \Gamma(D). \tag{4.4}$$

As D is integrable, Frobenius theorem ensures that M is foliated by leaves of D. Combining this fact with (4.4)*, we conclude that the leaves of D are totally geodesic submanifolds of M.*

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ON THE MEASURE OF TRANSCENDENCE OF $\zeta = \sum_{k=0}^{\infty} G_k^{-e_k}$ FORMAL LAURENT SERIES

AHMET Ş. ÖZDEMIR

ABSTRACT. In this work, we determine the transcendence measure of the formal Laurent series, $\varsigma \; = \; \sum\limits_{k=0}^{\infty} G_k^{-e^k}$ whose transcendence has been established by S. M. SPENCER [15]. Using the methods and lemmas in P. Bundschuh's article measure of transcendence for the above n is determined as

$$T(n, H) \; = \; H^{-(d+1)q^d - edq^{2d}}.$$

On the other hand, it was proven that transcendence series η is not a U but is a S or T numbers according to the Mahler's classification.

1. INTRODUCTION

Let p a prime number and $u \geq 1$ an integer. Let F be a finite field with $q = p^u$elements. We denote the ring of the polynomials with in one variable over F by $F[x]$and its quotient field by $F(x)$.If $a \in F[x]$is a non-zero polynomial, denote by ∂a its degree. If $a \; = \; 0$, then its degree is defined as $\partial 0 := -\infty$. Let a and b $(b \neq 0)$ two polynomials from $F[x]$ and define a discrete valuation of $F(x)$ as follows

$$\left| \frac{a}{b} \right| = q^{\partial a - \partial b}.$$

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Let K be the completion of $F(x)$ with respect to this valuation. Every element ω of K can be uniquely represented by

$$\omega = \sum_{n-k}^{\infty} c_b x^{-n}, c_n \in F.$$

If $\omega = 0$, then all c$_n$are zero. If $\omega \neq 0$, then there exist and $k \in Z$for which $c_k \neq 0$. If $\omega \neq 0$, then we have

$$|\omega| = q^{-k}.$$

Therefore K is the field all Formal Laurent series. The classical theory of transcendence over complex numbers has a similar version over K. Elements of $F[x]$ and $F(x)$ correspond to integers and fractions of the classical theory, respectively.

If $\omega$is one of the roots of a non-zero polynomial with coefficients in $F[x]$, then $\omega \in K$ is said to be algebraic over $F(x)$. Otherwise, $\omega$is called transcendental over $F(x)$.

The studies to find transcendental numbers in K were initiated first by Wade [16-19]. Also Geijsel [4-7] did similar studies. As it is the case in the classical theory of transcendental numbers, it is possible to define a measure of transcendence.

The measure of transcendence is thoroughly studied in the classical theory. For example, the transcendence measure of e has been widely investigated by Mahler [9], Fel'dman [3] and Cijsow [2]. Example for transcendence measure in the field K have been given for the first time by Bundschuh [1]. Further more, Özdemir showed the measure of transcendence of some Formal Laurent series [11],[12].

In this work, we determine the transcendence measure of some Formal Laurent series whose transcendence has been established by S.M.Spencer [15]. We take the $G_0 \left| G_1 \right| G_2 ..., d\ e\ g\ G_0 \geq 1, e = e_0 < e_1 < e_2 < ..., < e_k$ $\mid e_{k+1}, {}^{e_1}/_{e_2} \neq p'$for $r > s, e_k \in Z.$

If $G \in F[x]$ is a fixed non-zero polynomial of degree, $\partial(G_k) = g_k, g \geq 1$ then the series

$$\varsigma = \sum_{k=0}^{\infty} G_k^{-e_k} \tag{1}$$

is an element of K,and S.M.Spencer showed its transcendence in [14].

Using the methods and lemmas in Bundschuh's article [1], we determine a transcendence measure of ς. We take and arbitrary non-zero polynomial

$$P(y) = \sum_{v=0}^{n} a_v y^v, (a_v \in F[x]; v = 0, 1, ..., n) \tag{2}$$

Whose degree $\partial(P)$ is less than or equal to n. The height of P is denoted by

$$h(p) = \max_{v=0}^{n} |a_v| = q^{\max_{n\ v=0} \partial(a_v)}$$

For the transcendental element $\varsigma = \sum\limits_{k=0}^{\infty} G_k^{-e_k}$of K, we define the positive quantity

$$\Lambda_n(H, \varsigma) = \min |P(\varsigma)|\,,$$

where $P \not\equiv 0$, $\partial(P) \leq n, h(P) \leq H$. If $T(n,H)$ is a function of the variables n, H of $\Lambda_n(H,\varsigma)$ which satisfies the inequality

$$\Lambda_n(H,\varsigma) \geq T(n,H) \tag{3}$$

for all sufficiently large values of n and H, then $T(n,H)$ is said to be a transcendence measure of ς.

2. Preliminaries

Theorem 2.1. *We take an arbitrary, non-zero polynomial*

$$P(y) = \sum_{v=0}^{n} a_v y^v, (a_v \in F[x]; v = 0, 1, ..., n) \tag{4}$$

further let $\partial(P) = \ d, h(p) = h$ *and* $a = \max_{v=0}^{d} \partial a_v$.

$$dp^{mn} \log h \geq g_k e_k \log q. \tag{5}$$

Then we have

$$|P(\xi)| \geq h^{-(d+1)q^d - edq^{2d}} \quad (6) \tag{6}$$

and the transcendence measure of ω *is*

$$T(n,H) = \ H^{-(d+1)q^d - edq^{2d}} \quad (7) \tag{7}$$

As in the classical theory of transcendental number theory (see Schneider [13], Pagers 6), it is possible to define Mahler's classification on K. *Let* K *be transcendental, and define :*

$$\begin{aligned} \Theta\ _n(H,\eta) &:= \lim_{H \longrightarrow \infty} \sup \frac{-\log \Theta_n(H,\eta)}{\log H} \\ \Theta(\eta) &:= \lim_{n \to \infty} \sup \frac{1}{n} \Theta_n(\eta) \end{aligned} \tag{8}$$

Hence $\Theta_n(\eta) \geq n$ *for every* $n \in N$ *and so* $\Theta(\eta) \geq 1$. *For every* $n, H \in N$,

$$\Theta_n(H,\eta) < H^{-n} q^n \max(1, |\eta|^n) \quad (9) \tag{9}$$

is satisfied (see Bundschuh [1], Lemma 3).

On the other hand, let the least natural number n satisfying $\Theta_n(\eta) \geq \infty$ be donated by $\mu(\eta)$. If there is no such n, then on may define $\mu(\eta)$ as ∞. In this case, the transcendental number $\eta \in R$ is called

S-Laurent series if $1 \leq \Theta(\eta) < \infty$ and $\mu(\eta) = \infty$,

T-Laurent series if $\Theta(\eta) = \infty$ and $\mu(\eta) = \infty$,

U-Laurent series if $\Theta(\eta) = \infty$ and $\mu(\eta) < \infty$.

Moreover the U-class may be divided into subclasses. If $\mu(\eta) = m$ $(m > 0)$, then η is called a U_m- Laurent series. Le Vaque [8] was the first to show that for all m, U_m is non-empty in the classical theory but the honour goes to Oryan [10] if the ground field is K.

According to the above classification, the series defined in (1) can not be a U - Laurent series. This fact may be proved by the help of the Theorem 2.1.

Theorem 2.2. *The η Laurent series defined by (1) doesn't belong to the class U so that it belongs to the class S or to the class T.*

We will use the following lemmas in proof of the theorem.

Lemma 2.1. *Let*

$$P(y) = \sum_{v=0}^{n} a_v y^v$$

$$a_v \in F[x],\ a_d \neq 0\ (d \geq 1),\ a = \max_{v=0}^{d} \partial a_v \tag{10}$$

Then there are some elements $A_{0,}A_1, ..., A_d \in F[x]$, not all zero satisfying.

$$\partial A_1 \leq ad(q^d - d + 1) \quad \textit{for } 0 \leq j \leq d \textit{ and}$$

$$\sum_{j=0}^{d} A_j y^{q^j} = p(y) \sum_{j=0, q^j \geq d}^{d} A_j \sum_{k=0}^{q^j - d} b_k a_d^{-k-1} y^{q^j - d - k} =:, P(y)Q(y) \tag{11}$$

where $b_0 := 1$ and b_k, for $k \geq 1$ is the sum of product of exactly k terms from $a_0, a_1, ..., a_d$, multiplied by $(\pm)$.

Proof. See the [1], lemma 4, page 416.

Lemma 2.2. *Let $\eta \in K$ and $|\eta| = q^\lambda$. Under the hypotheses of Lemma 1 we have*

$$|Q(\eta)| \leq q^{ad(q^d - d + 1) + (q^d - d)\max(a, \lambda)}. \tag{12}$$

Proof. See the [1], lemma 5, page 417.

3. Proof of the Theorems

Proof. **(Theorem 1)**

Consider the polynomial defined by (4). With $\partial(p) = d, a_d \neq 0$. The Theorem is true obliviously for $d = 0$. Because then $|P(\eta)| = |a_0|$. $a_0 \in F[x]$ and since $a_0 \neq 0$ and we have, $|a_0| = q^{\partial(a_0)} > 1$. So the left side of (6) is less then 1. Let $d \geq 1$. By Lemma 1 there are some elements the $A_0, A_1, ..., A_d \in F[x]$ not all zero, such that

$$\sum_{j=0}^{d} A_j y^{q^j} = p(y) \sum_{j=0, q^j \geq d}^{d} A_j \sum_{k=0}^{q^j - d} b_k a_d^{-k-1} y^{q^j - d - k} =:, P(y)Q(y) \tag{13}$$

$$\partial A_j \leq ad(q^d - d + 1) \leq adq^d (0 \leq j \leq d) \tag{14}$$

In (13) we put η instead of y and using the fact that F is a field having q elements. We get

$$P(\eta)Q(\eta) = \sum_{j=0}^{d} A_j \eta^{q^j} = \sum_{j=0}^{d} A_j \sum_{k=0}^{\infty} G^{-e_k q^j}. \tag{15}$$

Separate the above sum as $S_1 + S_2$, where

$$S_1 = G^{e_\beta q^d} \sum_{j=0}^{d} A_j \sum_{k=0}^{+k_j} G^{-e_k q^j} \text{ and } S_2 = G^{e_\beta q^d} \sum_{j=0}^{d} A_j \sum_{k=k_j+1}^{\infty} G^{-e_k q^j} \tag{16}$$

where β is non-negative integer to be chosen later. Let the rational integers $k_j (j = 0, 1, ..., d)$ be defined by

$$q^{j-d} e_{k_j} < e_\beta \leq q^{d-j} e_{k_j+1} \tag{17}$$

1) First, we prove that $|S_1| \geq 1$. That is, we prove $S_1$is a polynomial bu not equal zero. Their terms of the S_1 are

$$G^{e_\beta q^d} A_j G^{-e_k q^j} = A_j G^{e_\beta q^d - e_k q^j} \tag{18}$$

We show that

$$e_\beta q^d - e_k q^j \geq 0 \tag{19}$$

by (17), and since k ranges from 0 to k_j in the sum S_1. We have

$$e_\beta q^d - e_k q^j \geq q^j (e_{k_j} - e_{k_j}) \geq 0 \tag{20}$$

which implies (19). By (19) and (18), S_1 is polynomial. Now we show S_1 isn't identically zero as equivalently. We have equality in (19) when and only when $k = \beta$ and $j = d$. If we write the terms of S_1, we find

$$S_1 = A_0 \left(\sum_{k=0}^{k_0} G^{e_\beta q^d - e_k q^0} \right) + ... + A_d \left(\sum_{k=0}^{k_d} G^{e_\beta q^d - e_k q^d} \right)$$

$$S_1 = A_0 \left(G^{e_\beta q^d - e_0 q^0} + ... + G^{e_\beta q^d - e_{k_0} q^0} \right) + ... + A_d \left(G^{e_\beta q^d - e_0 q^d} + ... + G^{e_\beta q^d - e_{k_d} q^d} \right) \tag{21}$$

$$\mu := \min_{j=0}^{d-1} (e_\beta q^d - e_{k_j} q^j, e_\beta q^d - e_{\beta-1} q^d) \tag{22}$$

G^μ divides of all terms in the sum(21) except only one term. Therefore ,

$$S_1 = G^\mu . R + A_d \quad (R \in F[x]) \tag{23}$$

and hence we find

$$S_1 \equiv A_d \, (\operatorname{mod} G^\mu) \tag{24}$$

Since $h = h(P) = q^a$,

$$a = \frac{\log h}{\log q} \tag{25}$$

By (5) and (25) we find

$$adq^d \geq \frac{g}{e} \tag{26}$$

From (19) and (26) it holds (27). For this. Consider the sequence

$$\{e_{-1}, e = e_0, e_1, e_2, ...\} .$$

There are β non-negative integers such that

$$e_{\beta-1} \leq \frac{adq^d}{g} < e_\beta \tag{27}$$

From (27) we obtain the following statement for the above β

$$\frac{adq^d}{g} < e_\beta \leq \frac{eadq}{g} \tag{28}$$

By (17) we have $e_\beta q^{d-j} \geq e_{k_j} \Longrightarrow q^{d-j} \geq \frac{e_{k_j}}{e_\beta} \Longrightarrow q^{d-j} - \frac{e_{k_j}}{e_\beta} \geq 0$. Hence we obtain

$$q^{d-j} - \frac{e_{k_j}}{e_\beta} \geq 1.(j < d) \tag{29}$$

further, since $e_{\beta-1} < e_\beta \Longrightarrow \frac{e_{\beta-1}}{e_\beta} < 1 \Longrightarrow 0 < 1 - \frac{e_{\beta-1}}{e_\beta}$. Thus we get

$$1 - \frac{e_{\beta-1}}{e_\beta} \geq 1 \tag{30}$$

From (22),

$$\mu = e_\beta \min_{j=0}^{d-1} q^j \left(\left(q^{d-j} - \frac{e_{k_j}}{e_\beta} \right), q^d \left(1 - \frac{e_{\beta-1}}{e_\beta} \right) \right) \tag{31}$$

by (29), (30) and (31) and q^q,$q^j > 1$ we get

$$\mu > e_\beta \tag{32}$$

by (14), (28) and (32) we obtain

$$g\mu > ge_\beta > adq^d > ad\left(q^d - d + 1\right) \geq \partial\left(A_d\right)$$

that is,

$$g\mu > \partial\left(A_d\right).$$

this inequality means

$$\partial\left(G^\mu\right) = g\mu > \partial\left(A_d\right).$$

Hence we see G^μ doesn't divide A_d. That is

$$A_d \not\equiv 0 (\operatorname{mod} G^\mu),$$

by (28) and (36)

$$S_1 \equiv A_d \not\equiv 0 (\operatorname{mod} G^\mu) \tag{33}$$

therefore S_1 is not identically 0. so S_1 is a non-zero polynomial. so it is shown that $|S_1| \geq 1$.

2) we will show $|S_2| < 1$ since $k \geq k_j + 1$ in S_2, for the degree of the terms of S_2, we may write the following inequality from (14):

$$\begin{aligned} \partial\left(G^{e_\beta q^d} A_j G^{-e_k q^j}\right) &= \partial A_j + \partial G^{e_\beta q^d - e_k q^j} \\ &\leq adq^d + g\left(e_\beta q^d - e_k q^j\right) \\ &\leq adq^d + g\left(e_\beta q^d - e_{k_j+1} q^j\right) \\ &\leq adq^d - ge_\beta\left(\frac{e_{k_j+1}}{e_\beta} q^j - q^d\right) \end{aligned} \tag{34}$$

by (17) $q^d e_\beta < q^j e_{k_j+1}$ $0 < \frac{e_{k_j+1}}{e_\beta} q^j - q^d$ is an integer. further, by (27) we obtain

$$adq^q < ge_\beta \tag{35}$$

from (34), (35) and since $\frac{e_{k_j+1}}{e_\beta}q^j - q^d$ is positive integer, we get

$$\partial\left(G^{e_\beta}A_jG^{-e_kq^j}\right) < 0$$

that is, the terms of S_2 have negative degrees. this means

$$|S_2| < 1$$

3) we will prove the claim of the theorem. by the definition of S_1 and S_2, we can write $S_1 + S_2 = G^{e_\beta q^d}P(\eta)Q(\eta)$. hence we obtain

$$|S_1 + S_2| = \left|G^{e_\beta q^d}\right| |P(\eta)| \, |Q(\eta)| \tag{36}$$

since $|S_1| \geq 1$ and $|S_2| < 1$, we get

$$|S_1 + S_2| = \max(|S_1|, |S_2|) = |S_1| \tag{37}$$

By (36) and (37), we obtain

$$|P(\eta)| \, |Q(\eta)| = |S_1| \left|G^{e_\beta q^d}\right|^{-1} \tag{38}$$

let $|\eta| = q^\lambda$. By (1) and since $|G^{se_k}| = q^{\deg G^{e_k}} = q^{ge_k}$,

we get $|\eta| = q^{-qe_0} = q^{-ge}$ therefore $\lambda = -ge$. since $\max(a, \lambda) = \max(a, -ge) = a$ and by lemma 2, we find

$$|Q(\eta)| \leq q^{ad(q^d-d+1)+(q^d-d)\max(a,\lambda)} \leq q^{adq^d+aq^d} \leq q^{a(d+1)q^d} \tag{39}$$

further, by (28)

$$\begin{aligned}\left|G^{e_\beta q^d}\right| &= q^{ge_\beta q^d} \\ &\leq q^{eadq^dq^d} \\ &= q^{eadq^{2d}}\end{aligned} \tag{40}$$

by (38),(39),(40) and since $|S_1| \geq 1$

$$\begin{aligned}|P(\eta)| &= |S_1| \left|G^{e_\beta q^d}\right|^{-1} |Q(\eta)|^{-1} \\ &\geq \left|G^{e_\beta q^d}\right|^{-1} |Q(\eta)|^{-1} \\ &\geq q^{eadq^{2d}} q^{-a(d+1)q^d}\end{aligned} \tag{41}$$

by (41) and since $h = q^a$

$$|P(\eta)| \geq h^{-(d+1)q^d - edq^{2d}}$$

this is the claim of the theorem 1.

Proof. **(Theorem 2)**

let the degree of the polynomial P in Theorem 1 be $\partial(P) = d \leq n$ and let its height be

$$h(P) = h \leq H \text{ by (6)},$$

$$|P(\eta)| \geq H^{-(n+1)q^n - enq^{2n}}. \tag{42}$$

(42) and (5) and by the definition of Mahler's classification

$$\Theta_n(H,\eta) \geq H^{-(n+1)q^n - enq^{2n}}$$

for all sufficiently large natural numbers n and H. hence consequently

$$\log \Theta_n(H,\eta) \geq \left[-(n+1)q^n - enq^{2n}\right] \log H$$

$$\frac{\log \Theta_n(H,\eta)}{\log H} \leq (n+1)q^n - enq^{2n} \tag{43}$$

$$\Theta_n(\eta) \geq \lim_{H\to\infty} \sup \frac{-\Theta_n(H,\eta)}{\log H} \leq enq^{2n} + (n+1)q^n \tag{44}$$

that is, for every index n

$$\Theta_n(\eta) < \infty$$

by the definition of Mahler's classification, $\mu(\eta) = \infty$. This shows η can never to the class U so that it belongs to the class S or to class T.

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TRANSLATION HYPERSURFACES WITH CONSTANT CURVATURE IN 4-DIMENSIONAL ISOTROPIC SPACE

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Abstract. In the present study, we deal with translation hypersurfaces in the 4-dimensional isotropic space $\mathbb{I}^4$ generated by translating planar curves. Due to absolute figure of $\mathbb{I}^4$ there are four different types of such hypersurfaces. We classify these translation hypersufaces in $\mathbb{I}^4$ with constant Gauss-Kronecker and mean curvature.

1. Introduction

Dillen et al. [8] introduced a *translation hypersurface* M^{n-1} in a $n-$dimensional Euclidean space $\mathbb{R}^n$ as the graph of the form

$$y_n = f_1(y_1) + ... + f_{n-1}(y_{n-1}), \tag{1.1}$$

where $(y_1, ..., y_n)$ denote orthogonal coordinates in $\mathbb{R}^n$ and $f_1, ..., f_n$ smooth functions of single variable. The authors in [8] proved that if M^{n-1} is minimal, it is either a hyperplane or $M^{n-1} = M^2 \times \mathbb{R}^{n-3}$, where M^2 is the *Scherk's minimal surface* (see [34]) given in explicit form

$$y_3 = \frac{1}{c} \ln \left| \frac{\cos(cy_2)}{\cos(cy_1)} \right|, \; c \in \mathbb{R}, \; c \neq 0.$$

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In many different ambient spaces, it was tried to generalize the Scherk's result as defining the translation (hyper)surfaces, see [7, 9, 13, 14, 18, 22, 23, 24, 38, 39, 41]. In addition, Seo [35] extended the above result to the translation hypersurfaces with arbitrary constant Gauss-Kronecker and mean curvature.

Munteanu et al. [28] brought forward a different perspective by generalizing the usual notion of translation surface and called it *translation graph.* More precisely, a *translation graph* in $\mathbb{R}^{p+q}$ is given in explicit form

$$y_{p+q}\left(y_1, y_2, ..., y_{p+q-1}\right) = f_1\left(y_1, ..., y_p\right) + f_2\left(y_{p+1}, ..., y_{p+q-1}\right),$$

for smooth functions $f_1 : \mathbb{R}^p \to \mathbb{R}$ and $f_2 : \mathbb{R}^{q-1} \to \mathbb{R}$. They provided certain minimality results on the translation graphs. In addition, Lima et al. [17] proved that a translation graph in $\mathbb{R}^{p+q}$ has vanishing Gauss-Kronecker curvature if it has nonzero constant Gauss-Kronecker or mean curvature.

Moruz and Munteanu [27] dealt with the minimal graphs of the form

$$y_4\left(y_1, y_2, y_3\right) = f_1\left(y_1\right) + f_2\left(y_2, y_3\right),$$

which can be expressed as the sum of a curve in y_1y_4-plane and a surface in $y_2y_3y_4-$space.

Notice that the graph of the form (1.1) is formed by translating $n-1$ curves (called *generating curves*) lying in mutually perpendicular 2-planes. This bring with two restrictions on the translation hypersurfaces: one is that generating curves are planar and the second that the planes including the generating curves are mutually perpendicular. As the restrictions are removed, the different kinds of the translation hypersurfaces arise. For example; in the particular case $n = 3$, Liu and Yu [19] introduced the notion of *affine translation surface,* i.e., the translation surface that the generating curves lie in non-perpendicular planes. They obtained minimal affine translation surfaces, so called *affine Scherk surfaces.* Furthermore, arbitrary constant mean curvature and Weingarten affine translation surfaces were presented in [15, 20].

In this study, we are interested in the counterparts of translation hypersurfaces in isotropic geometry, i.e., a particular Cayley-Klein geometry (for details, see [16, 29, 40]). In 3-dimensional isotropic space $\mathbb{I}^3$, if the generating curves are chosen to lie in mutually perpendicular planes, then three types of translation surfaces exist due to the absolute figure. Let M^2 denote a translation surface in $\mathbb{I}^3$, then we have

Type 1. both generating curves lie in isotropic planes; that is, M^2 is a graph of the form

$$x_3(x_1, x_2) = f(x_1) + g(x_2),$$

where (x_1, x_2, x_3) denote the isotropically orthogonal coordinates in $\mathbb{I}^3$.

Type 2. One generating curve lies in non-isotropic plane and other in isotropic plane; that is, M^2 is a graph of the form

$$x_2(x_1, x_3) = f(x_1) + g(x_3).$$

Type 3. Both generating curves lie in non-isotropic planes; that is, M^2 is a graph of the form

$$x_1(x_2, x_3) = \frac{1}{2}\left(f(x_2 + x_3 - \pi/2) + g(\pi/2 - x_2 + x_3)\right).$$

As well as the non-isotropic planes, Strubecker [36] obtained the minimal translation surfaces in $\mathbb{I}^3$, so called *isotropic Scherk's surfaces of type 1,2,3.* These surfaces are respectively given as follows: for $c \in \mathbb{R},\ c \neq 0$, $x_3 = c\left(x_1^2 - x_2^2\right)$ $c \in \mathbb{R}$, $c \neq 0$ (type 1),

$$x_2 = \frac{1}{c}\ln\left|\frac{cx_3}{\cos cx_1}\right| \text{ (type 2) and } x_1 = \frac{1}{2c}\ln\left|\frac{\cos c(x_2 + x_3 - \pi/2)}{\cos c(\pi/2 - x_2 + x_3)}\right| \text{ (type 3).}$$

Recently, these results were generalized by Milin-Sipus [25] to the translation surfaces in $\mathbb{I}^3$ with arbitrary constant Gaussian and mean curvature. The situation that the generating curves in $\mathbb{I}^3$ are non-planar extends the above categorization and the results. For example, see [1, 4].

In $\mathbb{I}^4$, there are four types of translation hypersurfaces whose the generating curves lie in mutually perpendicular $k-$planes $(k = 2, 3)$, see Section 3. In more general case, i.e. in arbitrary dimensional isotropic spaces, the translation hypersurfaces of type 1 were studied in [3]. The present study concerns other three types of translation hypersurfaces in $\mathbb{I}^4$ with constant Gauss-Kronecker and mean curvature.

Due to the absolute figure of $\mathbb{I}^n$ $n \geq 3$, for a smooth real-valued function f the graph hypersurfaces associated with the form $x_n = f(x_1, ..., x_{n-1})$ differ from other hypersurfaces. For example; the Gauss-Kronecker and mean curvature for such a graph hypersurface in $\mathbb{I}^n$ correspond to determinant and the trace of the Hessian of f, respectively. The formulas of these curvatures were provided by Chen et al. [6]. As far as we know, this is first study formulating such fundamental curvatures for a generic hypersurface in $\mathbb{I}^n$.

2. Preliminaries

Some differential geometric approaches on curves and hypersurfaces in isotropic geometry can be found in [2, 5, 10, 12, 21, 26, 11, 30, 31, 32, 33].

Let $\mathbb{P}^n$ denote the $n-$dimensional real projective space, ω a hyperplane in $\mathbb{P}^n$ and $\mathbb{I}^n = \mathbb{P}^n \backslash \omega$ the obtained affine space. We call $\mathbb{I}^n$ *$n-$dimensional isotropic space* if ω contains a hypersphere $\mathbb{S}$ with null radius. Then the pair $\{\omega, \mathbb{S}\}$ is called *absolute figure* of $\mathbb{I}^n$ and parametrized in homogeneous coordinates by

$$\omega : u_0 = 0,\ \mathbb{S} : u_0 = u_1^2 + ... + u_{n-1}^2 = 0.$$

The vertex of $\mathbb{S}$ is $F\,(0 : 0 : ... : 1)$ called *absolute point*. Here, by a *vertex* we mean the intersection of all maximal generators of a quadric. For more details, see [37].

Denote affine coordinates $x_1 = \frac{u_1}{u_0}, ..., x_n = \frac{u_n}{u_0}$, $u_0 \neq 0$. Then the group of motions of $\mathbb{I}^n$ which preserves the absolute figure is given in terms of affine coordinates by

$$\begin{bmatrix} A & 0 \\ B & 1 \end{bmatrix},$$

where A is an orthonogal $(n-1, n-1)-$matrix, B a real $(1, n-1)-$matrix.

Let $p = (p_1, ..., p_n)$, $q = (q_1, ..., q_n)$ be two points in $\mathbb{I}^n$. The *isotropic distance* between p and q is defined by

$$d_i\,(p, q) = \sqrt{\sum_{i=1}^{n-1} (p_i - q_i)^2}.$$

If $d_i = 0$, then the so-called *range* between p and q is defined as $d_i^r = |p_n - q_n|$.

A line is said to be *isotropic* if its point at infinity is absolute. Other lines are *non-isotropic*. We call a $k-$plane *isotropic* (*non-isotropic*) if it contains (does not) an isotropic line. In the affine model of $\mathbb{I}^n$, the isotropic lines and the isotropic $k-$planes are parallel to x_n-axis. For example; the following

$$a_1x_1 + ... + a_nx_n = b,\ a_i, b \in \mathbb{R},$$

determines an isotropic (non-isotropic) hyperplane if $a_n = 0\ (\neq 0)$.

Note that the hyperplane $x_n = 0$, so-called *basic hyperplane*, is non-isotropic and therefore the Euclidean metric is used in it.

As distinct from the Euclidean case, the orthogonality in $\mathbb{I}^n$ does not bring with the perpendicularity. Obviously, two non-isotropic lines are orthogonal if their projections onto the basic hyperplane are perpendicular up to the Euclidean metric. Nevertheless, an isotropic

line is orthogonal to some non-isotropic line. As a consequence, each non-isotropic hyperplane is orthogonal to the isotropic one. In addition, two isotropic hyperplanes are orthogonal if their projections onto the basic hyperplane are perpendicular.

We call a curve *isotropic* (*non-isotropic*) $k-$*planar* if it lies in an isotropic (non-isotropic) $k-$plane.

2.1. **Curvature theory of hypersurfaces.** This part of isotropic geometry is similar to the Euclidean case.

Let M^{n-1}, $n \geq 3$, be a hypersurface in $\mathbb{I}^n$ whose the tangent hyperplane at each point is non-isotropic. Such a hypersurface is said to be *admissible.* Then the coefficients g_{ij} of the first fundamental form are calculated by the induced metric from $\mathbb{I}^n$. The normal vector field U of M^{n-1} is completely isotropic, i.e. $(0,0,...,1)$.

For the second fundamental form, let us consider a curve r on M^{n-1} with isotropic arclength s and the tangent vector $t(s) = r'(s) = \frac{dr}{ds}$. Denote S the projection of $r''(s) = \frac{d^2r}{ds^2}$ onto the tangent hyperplane of M^{n-1}. Then, the following decomposition occurs:

$$r''(s) = \kappa_g S + \kappa_n U,$$

where κ_g and κ_n are *geodesic* and *normal curvatures* of r, respectively. Hence, it follows $\kappa_g = \|r''(s)\|_i$, where $\|\cdot\|_i$ indicates the isotropic norm. In addition, by a direct computation, we have

$$\kappa_n = \frac{1}{\sqrt{\det g_{ij}}} \sum_{i,j=1}^{n-1} \det\left(r_{x_1},...,r_{x_{n-1}},r_{x_ix_j}\right) \frac{dx_i}{ds}\frac{dx_j}{ds}, \tag{2.1}$$

where $r_{x_i} = \frac{\partial r}{\partial x_i}$ and $r_{x_ix_j} = \frac{\partial^2 r}{\partial x_i \partial x_j}$, $1 \leq i,j \leq n-1$. If we put

$$h_{ij} = \frac{\det\left(r_{x_1},...,r_{x_{n-1}},r_{x_ix_j}\right)}{\sqrt{\det g_{ij}}}$$

into (2.1) then one can be rewritten in the matrix form as

$$\kappa_n = \tilde{t}^T \cdot [h_{ij}] \cdot \tilde{t},\ \tilde{t} = \left(\frac{dx_1}{ds},...,\frac{dx_{n-1}}{ds}\right)^T, \tag{2.2}$$

where "$\cdot$" denotes the matrix multiplication. If r is a curve with arbitrary parameter, then (2.2) turns to

$$\kappa_n = \frac{\tilde{t}^T \cdot [h_{ij}] \cdot \tilde{t}}{\tilde{t}^T \cdot [g_{ij}] \cdot \tilde{t}}.$$

The extreme values of κ_n, which we call *principal curvatures,* correspond to the eigenvalues of the matrix $[h_{ij}] \cdot [g_{ij}]^{-1}$. Let us denote the principal curvatures $\kappa_1,...,\kappa_{n-1}$ and put $[a_{ij}] =$

$[h_{ij}] \cdot [g_{ij}]^{-1}$. Therefore, the characteristic equation of $[a_{ij}]$ follows

$$\det([a_{ij}] - \lambda I_{n-1}) = \lambda^{n-1} - tr[a_{ij}]\lambda^{n-2} + ... + (-1)^{n-1}\det[a_{ij}] = 0,$$

which provides the fundamental curvatures, called *isotropic Gauss-Kronecker curvature* (or *relative curvature*) and *isotropic mean curvature*. We shortly call them *Gauss-Kronecker* (K) and *mean curvature* (H). Obviously, one obtains

$$K = \kappa_1 ... \kappa_{n-1} = \det[a_{ij}] \text{ or } K = \frac{\det[h_{ij}]}{\det[g_{ij}]}$$

and

$$(n-1)H = \kappa_1 + ... + \kappa_{n-1} = tr[a_{ij}],$$

where tr denotes the trace of a matrix.

A hypersurface is said to be flat (minimal) if K (H) is identically zero.

Notice that the isotropic counterpart for the notion of *shape operator* in the Euclidean sense of a hypersurface is indeed a zero map. The matrix $[a_{ij}]$ however plays the role of the matrix corresponding shape operator in $\mathbb{I}^n$.

3. Categorization of translation hypersurfaces

Let M^3 be a translation hypersurface in $\mathbb{I}^4$ generated by translating three curves lying in mutually perpendicular $k-$planes, $k = 2, 3$. Denote the generating curves α, β, γ. Up to the absolute figure of $\mathbb{I}^4$ there are four types of such hypersurfaces given as follows:

Type 1. Three of α, β, γ are isotropic 2-planar. Then M^3 is parameterized by

$$r(u, v, w) = (u, v, w, f(u) + g(v) + h(w)),$$

where α, β and γ lie in x_1x_4-plane, x_2x_4-plane and x_3x_4-plane, respectively.

Type 2. α is non-isotropic 2-planar and β, γ isotropic 2-planar. Then M^3 is parameterized by

$$r(u, v, w) = (u + v, w, f(u), g(v) + h(w)),$$

where α, β and γ lie in x_1x_3-plane, x_1x_4-plane and x_2x_4-plane, respectively. Admissibility implies that f is a non-constant function.

Type 3. α, β are non-isotropic 2-planar and γ isotropic 2-planar. Then M^3 is parameterized by

$$r(u, v, w) = (u + v + w, f(u), g(v), h(w)),$$

where α, β and γ lie in x_1x_2-plane, x_1x_3-plane and x_1x_4-plane, respectively. Admissibility implies that neither f nor g is a constant function.

Type 4. Three of α, β, γ are non-isotropic hyperplanar. The curves α, β, γ and the hyperplanes $P_\alpha, P_\beta, P_\gamma$ containing them can be choosen as

$$\alpha(u) = (f(u), u, u, u+\pi), \ P_\alpha : -2x_2 + x_3 + x_4 = \pi;$$
$$\beta(v) = \left(g(v), v, v, -v + \tfrac{\pi}{3}\right), \ P_\beta : 2x_2 + x_3 + 3x_4 = \pi;$$
$$\gamma(w) = \left(h(w), 6w, -w, w - \tfrac{\pi}{2}\right), \ P_\gamma : x_2 + 4x_3 - 2x_4 = \pi.$$

Then M^3 is parameterized by

$$r(u,v,w) = \left(f(u) + g(v) + h(w), u + v + 6w, u + v - w, u - v + w + \frac{5\pi}{6}\right),$$

where $\frac{df}{du} - \frac{dg}{dv} \neq 0$ because admissibility.

A translation hypersurface of above one type is no equivalent to that of other type due to the absolute figure of $\mathbb{I}^4$.

We hereinafter denote the derivatives of f, g, h with respect to the given variable by a prime and so.

4. Translation hypersurfaces of type 2

For a translation hypersurface of type 2, the matrices of the fundemantal forms are given by

$$[g_{ij}] = \begin{pmatrix} 1+(f')^2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [g_{ij}]^{-1} = \frac{1}{(f')^2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1+(f')^2 & 0 \\ 0 & 0 & (f')^2 \end{pmatrix}$$

and

$$[h_{ij}] = \begin{pmatrix} -\frac{f''g'}{f'} & 0 & 0 \\ 0 & -g'' & 0 \\ 0 & 0 & -h'' \end{pmatrix}.$$

Hence the Gauss-Kronecker and the mean curvature follows respectively

$$K = \frac{g'f''g''h''}{(f')^3} \tag{4.1}$$

and

$$3H = \frac{f''g'}{(f')^3} + g''\frac{1+(f')^2}{(f')^2} + h''. \tag{4.2}$$

Theorem 4.1. *Let M^3 be a flat translation hypersurface of type 2 in $\mathbb{I}^4$. Then it is a cylindrical hypersurface with non-isotropic rulings. Furthermore if M^3 has nonzero constant*

Gauss-Kronecker curvature then, up to suitable constants and translations of u, v, w, *the following holds*

$$f(u) = \lambda u^{\frac{1}{2}}, \ g(v) = \mu v^{\frac{3}{2}}, \ h(w) = \xi w^2,$$

where $\lambda, \mu, \xi \in \mathbb{R}$ *and* $\lambda\mu\xi \neq 0$.

Proof. The (4.1) follows that K vanishes if at least one of f, g, h is a linear function with respect to the given variable, that is, at least one of the generating curves turns to a non-isotropic line. Without loss of generality we may assume that f is linear, i.e. $f(u) = c_1 u + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. Hence, one can be parameterized by

$$r(u, v, w) = u(1, 0, c_1, 0) + (v, w, c_2, g(v) + h(w)),$$

which means that M^3 is congruent a cylindrical hypersurface with non-isotropic rulings. Now, let assume that the Gauss-Kronecker curvature is a nonzero constant K_0. So, the (4.1) leads to

$$\frac{f''}{(f')^3} = c_3, \ g'g'' = c_4, \ h'' = c_5, \tag{4.3}$$

for $c_3, c_4, c_5 \in \mathbb{R}$ and $K_0 = c_3 c_4 c_5 \neq 0$. After solving (4.3), we obtain

$$f(u) = \pm\frac{1}{c_3}\sqrt{-2c_3 u + c_6} + c_7, \quad g(v) = \pm\frac{1}{3c_4}(2c_4 v + c_8)^{\frac{3}{2}} + c_9$$

and

$$h(w) = \frac{c_5}{2}w^2 + c_{10}w + c_{11},$$

where $c_6, ..., c_{11} \in \mathbb{R}$. Up to congruency of $\mathbb{I}^4$one may assume $c_7 = c_9 = c_{11} = 0$ and up to a translation on u, v, wchoose $c_6 = c_8 = c_{10} = 0$. Besides putting $\lambda = \frac{\pm\sqrt{-2c_3}}{c_3}$, $\mu = \frac{\pm(2c_4)^{\frac{3}{2}}}{3c_4}$ and $\xi = \frac{c_5}{2}$ completes the proof.

Theorem 4.2. *Let* M^3 *be a minimal translation hypersurface of type 2 in* $\mathbb{I}^4$. *Then* M^3 *is either a non-isotropic hyperplane or, up to suitable constants and translations of* u, v, w, *one of the following cases occurs:*

(i) $f = f(u)$, $f' \neq 0$, $g(v) = \lambda$, $h(w) = \mu w$;
(ii) $f(u) = \lambda u^{\frac{1}{2}}$, $g(v) = \mu v$, $h(w) = \frac{\mu}{\lambda^2}w^2$, $\lambda\mu \neq 0$;
(iii) $f(u) = \lambda u$, $g(v) = \mu v^2$, $h(w) = -\frac{1+\lambda^2}{\lambda^2}\mu w^2$, $\lambda\mu \neq 0$;
(iv) $M^3 = S^2 \times \mathbb{R}$, *where* S^2 *is the isotropic Scherk's surface of type 2 in* $\mathbb{I}^3$,

where $\lambda, \mu, \xi \in \mathbb{R}$.

Proof. The (4.2) leads to

$$\frac{f''g'}{(f')^3} + \frac{1+(f')^2}{(f')^2}g'' + h'' = 0. \tag{4.4}$$

The partial derivative of (4.4) with respect to w implies $h'' = h_0$, $h_0 \in \mathbb{R}$. If $g' = 0$, we get $h(w) = c_1 w + c_2$. Putting $c_1 = \mu$ and applying a translation on w implies that M^3 is congruent to the hypersurface given in the case (i) of the theorem. Afterwards we assume $g' \neq 0$. Then the partial derivative of (4.4) with respect to v yields

$$\frac{f''}{f'}g'' + \left[1+(f')^2\right]g''' = 0. \tag{4.5}$$

If g'' and g''' are linearly independent then the contradiction $1+(f')^2 = 0$ is obtained. Hence we have either $g'' = 0$ or $g''' = kg''$, $g'' \neq 0$ and $k \in \mathbb{R}$.

(1) $g'' = 0$. (4.4) can be rewritten by putting $g' = g_0 \neq 0$ as

$$\frac{f''}{(f')^3}g_0 + h_0 = 0. \tag{4.6}$$

Being $f'' = 0 = h_0$ is a solution to (4.6), which leads M^3 to be a non-isotropic hyperplane. If $f''h_0 \neq 0$, (4.6) turns to

$$\frac{f''}{(f')^3} = -\frac{h_0}{g_0}. \tag{4.7}$$

By solving (4.7), we derive

$$f(u) = \pm\frac{g_0}{h_0}\sqrt{2\frac{h_0}{g_0}u + c_3} + c_4, \quad g(v) = g_0 v + c_5$$

and

$$h(w) = \frac{h_0}{2}w^2 + c_6 w + c_7.$$

Up to congruency of $\mathbb{I}^4$ one may assume $c_4 = c_7 = 0$ and up to a translation on u, v, w choose $c_3 = c_5 = c_6 = 0$. After putting $\lambda = \pm\frac{g_0}{h_0}\sqrt{2\frac{h_0}{g_0}}$ and $\mu = g_0$ we obtain that M^3 is congruent to the hypersurface given in the case (ii) of the theorem.

(2) $g''' = kg''$, $g'' \neq 0$. (4.5) leads to

$$f'' = -kf'\left[1+(f')^2\right]. \tag{4.8}$$

Being $f'' = 0 = k$ is a solution for (4.8). Therefore we write

$$f(u) = c_8 u + c_9, \; g(v) = \frac{c_{10}}{2}v^2 + c_{11}v + c_{12},$$

where $c_8, ..., c_{12} \in \mathbb{R}$, $c_8 c_{10} \neq 0$. Considering it into (4.4) concludes $h'' = -\frac{1+c_8^2}{c_8^2} c_{11}$ or

$$h(w) = -\frac{1+c_8^2}{c_8^2} c_{10} w^2 + c_{13} w + c_{14},$$

for $c_{13}, c_{14} \in \mathbb{R}$. As in the previous cases, up to suitable constants and translations, we obtain that M^3 is congruent to the hypersurface given in the case (iii) of the theorem. Assuming $k \neq 0$ in (4.8) yields $f'' \neq 0$. Also we have $g'' = kg' + c_{15}$, $l \in \mathbb{R}$ by integrating $g''' = kg''$. Hence substituting (4.8) into (4.4) gives

$$c_{15} \frac{1 + (f')^2}{(f')^2} + h_0 = 0. \tag{4.9}$$

The admissibility implies that f is a non-constant function and thus we conclude from (4.9) that $c_{15} = h_0 = 0$, i.e. $h(w) = c_{16} w + c_{17}$ for $c_{16}, c_{17} \in \mathbb{R}$. Because (4.8) and being $g'' = kg'$, we write

$$\frac{f''}{f'\left[1 + (f')^2\right]} = -k = -\frac{g''}{g'}. \tag{4.10}$$

After solving (4.10), we obtain

$$f(u) = \pm \frac{1}{k} \arccos\left(c_{18} e^{-ku}\right), \; g(v) = -\frac{c_{19}}{k} e^{kv}, \tag{4.11}$$

for $c_{18}, c_{19} \in \mathbb{R}$, $c_{18} c_{19} \neq 0$. By a change of parameter in (4.11) M^3 can be parameterized as

$$r(\tilde{u}, \tilde{v}, w) = \left(\frac{1}{k} \ln \left| \frac{k\tilde{v}}{\cos k\tilde{u}} \right|, 0, \tilde{u}, \tilde{v} \right) + w(0, 1, 0, c_{16})$$

and is congruent to $S^2 \times \mathbb{R}$, where S^2 is the isotropic Scherk's surface of type 2 in $\mathbb{I}^3$. This completes the proof.

Theorem 4.2. immediately implies the following corollary

Corollary 4.1. *Let M^3 be a translation hypersurface of type 2 in $\mathbb{I}^4$. Then, $H = 0$ implies $K = 0$.*

Theorem 4.3. *Let M^3 be a translation hypersurface of type 2 in $\mathbb{I}^4$ with nonzero constant mean curvature H_0. Then, up to suitable constants and translations of u, v, w, one of the following cases occurs:*

(i) $f = f(u)$, $f' \neq 0$, $g(v) = \lambda$, $h(w) = \frac{3H_0}{2} w^2$;

(ii) $f(u) = \lambda u$, $g(v) = \mu v$, $h(w) = \frac{3H_0}{2} w^2$, $\lambda\mu \neq 0$;

(iii) $f(u) = \lambda u^{\frac{1}{2}}$, $g(v) = \mu v$, $h(w) = \xi w^2$, $\lambda\mu \neq 0$, $3H_0 = \frac{-2\mu}{\lambda^2} + 2\xi \neq 0$;

(iv) $f(u) = \lambda u$, $g(v) = \mu v^2$, $h(w) = \xi w^2$, $\lambda\mu \neq 0$, $3H_0 = \frac{2\mu(1+\lambda^2)}{\lambda^2} + 2\xi \neq 0$;

(v) $M^3 = S^2 \times P$, *where* S^2 *is the isotropic Scherk's surface of type 2 in* $\mathbb{I}^3$ *and* P *is a parabolic circle in* $\mathbb{I}^2$ *with isotropic curvature* $3H_0$,

where $\lambda, \mu, \xi \in \mathbb{R}$.

Proof. Reconsidering (4.2) leads to $h'' = h_0$, $h_0 \in \mathbb{R}$ and therefore we get

$$3H_0 = \frac{f''g'}{(f')^3} + g''\frac{1+(f')^2}{(f')^2} + h_0. \tag{4.12}$$

To solve (4.12), we distinguish two cases:

(1) $g' = g_0$, $g_0 \in \mathbb{R}$. In particular; if $g_0 = 0$, then we conclude $h_0 = 3H_0$ and

$$h(w) = \frac{3}{2}H_0 w^2 + c_1 w + c_2, \ c_1, c_2 \in \mathbb{R},$$

which implies that M^3 is congruent to the hypersurface given in the case (i) of the theorem. Nevertheless; if $g_0 \neq 0$ then, by (4.12) we get

$$\frac{3H_0 - h_0}{g_0} = \frac{f''}{(f')^3}. \tag{4.13}$$

If $3H_0 = h_0$ in (4.13), we immediately obtain the proof the case (ii) of the theorem. Otherwise, after solving (4.13) we obtain

$$f(u) = \pm\frac{g_0}{3H_0 - h_0}\sqrt{\frac{-6H_0 + 2h_0}{g_0}u + c_3} + c_4,$$

where $3H_0 \neq h_0$ and $c_3, c_4 \in \mathbb{R}$. Hence, after suitable translations and constants, we obtain that M^3 is congruent to the hypersurface given in the case (iii) of the theorem.

(2) $g'' \neq 0$. We consider two cases:

(a) $f' = f_0 \neq 0$, $f_0 \in \mathbb{R}$. (4.12) leads to

$$3H_0 = \frac{1+f_0^2}{f_0^2}g'' + h_0,$$

which implies the proof of the case (iv) of the theorem up to constants and suitable translations.

(b) $f'' \neq 0$. (4.12) implies $h_0 = 3H_0$ and

$$\frac{f''}{(f')^3} = c_5\frac{1+(f')^2}{(f')^2}, \quad g'' = -c_5 g', \tag{4.14}$$

where $c_5 \in \mathbb{R}$, $c_5 \neq 0$. After solving (4.14), we obtain

$$f(u) = \pm\frac{1}{\lambda}\arccos\left(c_6 e^{c_5 u}\right), \ g(v) = -\frac{c_7}{\lambda}e^{-c_5 v} \tag{4.15}$$

for $c_6, c_7 \in \mathbb{R}$, $c_6 c_7 \neq 0$. By a change of parameter in (4.15), M^3 can be written as

$$r(\tilde{u}, \tilde{v}, w) = \left(\frac{1}{\lambda} \ln \left|\frac{\cos \lambda \tilde{u}}{\lambda \tilde{v}}\right|, 0, \tilde{u}, \tilde{v}\right) + \left(0, w, 0, \frac{3}{2} H_0 w^2\right),$$

which completes the proof of the theorem.

5. Translation hypersurfaces of type 3

For a translation hypersurface of type 3, the matrices of the fundemantal forms are given by

$$[g_{ij}] = \begin{pmatrix} 1+(f')^2 & 1 & 1 \\ 1 & 1+(g')^2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad [g_{ij}]^{-1} = \begin{pmatrix} \frac{1}{(f')^2} & 0 & \frac{-1}{(f')^2} \\ 0 & \frac{1}{(g')^2} & \frac{-1}{(g')^2} \\ \frac{-1}{(f')^2} & \frac{-1}{(g')^2} & 1+\frac{1}{(f')^2}+\frac{1}{(g')^2} \end{pmatrix}$$

and

$$[h_{ij}] = \begin{pmatrix} \frac{f''h'}{f'} & 0 & 0 \\ 0 & \frac{g''h'}{g'} & 0 \\ 0 & 0 & h'' \end{pmatrix}.$$

Hence the Gauss-Kronecker and the mean curvature are respectively

$$K = \frac{(h')^2 f'' g'' h''}{(f'g')^3} \tag{5.1}$$

and

$$3H = h'\left[\frac{f''}{(f')^3} + \frac{g''}{(g')^3}\right] + h''\left[1 + \frac{1}{(f')^2} + \frac{1}{(g')^2}\right]. \tag{5.2}$$

Theorem 5.1. *Let M^3 be a flat translation hypersurface of type 3 in $\mathbb{I}^4$. Then it is a cylindrical hypersurface with non-isotropic rulings. Furthermore if M^3 has nonzero constant Gauss-Kronecker curvature then, up to suitable constants and translations of u, v, w, the following holds*

$$f(u) = \lambda u^{\frac{1}{2}}, \ g(v) = \mu v^{\frac{1}{2}}, \ h(w) = \xi w^{\frac{4}{3}},$$

where $\lambda, \mu, \xi \in \mathbb{R}$ and $\lambda\mu\xi \neq 0$.

Proof. The (5.1) follows that K vanishes if at least one of f, g, h is a linear function with respect to the given variable; that is, at least one of the generating curves turns to be a non-isotropic line. Without loss of generality we may assume that f is linear, i.e. $f(u) = c_1 u + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. Hence, M^3 can be parameterized by

$$r(u, v, w) = u(1, c_1, 0, 0) + (v + w, c_2, g(v) + h(w)),$$

which means that it is congruent to a cylindrical hypersurface with non-isotropic rulings. Now, let us assume that K is a nonzero constant. So, (5.1) leads to

$$\frac{f''}{(f')^3} = c_3, \quad \frac{g''}{(g')^3} = c_4, \quad (h')^2 h'' = c_5, \tag{5.3}$$

for $c_3, c_4, c_5 \in \mathbb{R}$ and $c_3 c_4 c_5 \neq 0$. After solving (5.3), we obtain

$$f(u) = \pm \frac{1}{c_3} \sqrt{-2c_3 u + c_6} + c_7, \; g(v) = \pm \frac{1}{c_4} \sqrt{-2c_4 v + c_8} + c_9$$

and

$$h(w) = \frac{1}{4c_5} (3c_5 w + c_{10})^{\frac{4}{3}} + c_{11},$$

where $c_6, ..., c_{11} \in \mathbb{R}$. Up to congruency of $\mathbb{I}^4$one may assume $c_7 = c_9 = c_{11} = 0$ and up to a translation on u, v, wwe choose $c_6 = c_8 = c_{10} = 0$. Eventually, putting $\lambda = \frac{\pm\sqrt{-2c_3}}{c_3}$, $\mu = \frac{\pm\sqrt{-2c_4}}{c_4}$ and $\xi = \frac{(3c_5)^{\frac{4}{3}}}{4c_5}$ completes the proof.

Theorem 5.2. *Let M^3 be a minimal translation hypersurface of type 3 in $\mathbb{I}^4$. Then, M^3 is either a non-isotropic hyperplane or, up to suitable constants and translations of u, v, w, one of the following cases occurs:*

(i) $f = f(u)$, $f' \neq 0$, $g = g(v)$, $g' \neq 0$, $h(w) = \lambda$;
(ii) $f(u) = \lambda(-u)^{\frac{1}{2}}$, $g(v) = \lambda v^{\frac{1}{2}}$, $h(w) = \mu w$, $\lambda\mu \neq 0$;
(iii) $M^3 = S^2 \times \mathbb{R}$, *where S^2 is the isotropic Scherk' s surface of type 2 in $\mathbb{I}^3$;*
(iv) $f(u) = \eta \ln \left| \frac{1+\sqrt{1+\kappa e^{\lambda u}}}{1-\sqrt{1+\kappa e^{\lambda u}}} \right|$ *or* $f(u) = \kappa e^{\lambda u}$, $g(v) = \mu \ln \left| \frac{1+\sqrt{1+\xi e^{\varpi v}}}{1-\sqrt{1+\xi e^{\varpi v}}} \right|$ *or* $g(v) = \xi e^{\varpi v}$, $h(w) = \rho e^{\tau w}$, *where $\eta, \kappa, \lambda, \mu, \xi, \varpi, \rho, \tau$ are nonzero constants.*

Proof. Due to $H = 0$, (5.2) reduces to

$$h' \left[\frac{f''}{(f')^3} + \frac{g''}{(g')^3} \right] + h'' \left[1 + \frac{1}{(f')^2} + \frac{1}{(g')^2} \right] = 0. \tag{5.4}$$

It immediately follows from (5.4) that h' and h'' can not be linearly independent. In this manner we have either $h' = 0$ or $h'' = c_1 h'$, $h' \neq 0$ and $c_1 \in \mathbb{R}$. Being $h' = 0$ implies that M^3 is congruent to the hypersurface given in the first case of the theorem. Now we assume that $h'' = c_1 h'$ and $h' \neq 0$. To solve (5.4) there are two cases:

(1) $c_1 = 0$, i.e. $h(w) = c_2 w + c_3$, $c_2, c_3 \in \mathbb{R}$, $c_2 \neq 0$. Hence (5.4) reduces to

$$\frac{f''}{(f')^3} = c_4 = \frac{-g''}{(g')^3}, \; c_4 \in \mathbb{R}. \tag{5.5}$$

If $c_4 = 0$, then M^3 turns to be a non-isotropic hyperplane. Otherwise, i.e. $c_4 \neq 0$, solving (5.5) leads to

$$f(u) = \pm \frac{1}{c_4}\sqrt{-2c_4 u + c_5} + c_6,\ g(v) = \pm \frac{1}{c_4}\sqrt{2c_4 v + c_7} + c_8,$$

where $c_6, ..., c_8 \in \mathbb{R}$. Up to congruency of $\mathbb{I}^4$ one may assume $c_6 = c_8 = 0$. We may aslo choose $c_3 = c_5 = c_7 = 0$ up to a translation on u, v, w. By putting $\lambda = \frac{\pm\sqrt{2c_4}}{c_4}$ and $\mu = c_3$, we obtain that M^3 is congruent to the hypersurface given in the case (ii) of the theorem.

(2) $c_1 \neq 0$. Then (5.4) yields

$$\frac{f''}{(f')^3} + \frac{c_1}{(f')^2} + \frac{g''}{(g')^3} + \frac{c_1}{(g')^2} = -c_1, \tag{5.6}$$

where the roles of f and g are symmetric and thus it is enough to discuss the situation on f. We have two cases:

(a) $f' = f_0 \in \mathbb{R}$. (5.6) implies

$$\frac{f_0^2 g''}{g'\left[\left(1+f_0^2\right)(g')^2 + f_0^2\right]} = -c_1. \tag{5.7}$$

After solving (5.7), we obtain

$$g(v) = \pm \frac{f_0}{\sqrt{1+f_0^2}} \arccos\left(c_9\left[1+f_0^2\right]e^{-c_1 v}\right),\ c_9 \in \mathbb{R},\ c_9 \neq 0.$$

On the other hand, since $h'' = c_1 h'$ we get $h(w) = c_{10}e^{c_1 w}$, $c_{10} \in \mathbb{R}$, $c_{10} \neq 0$. By a change of parameter and up to suitable constants and translations we derive that M^3 is congruent to the hypersurface given in the case (iii) of the theorem.

(b) $f'' \neq 0$. By symmetry, we have $g'' \neq 0$. Thereby, (5.6) implies

$$\frac{f''}{(f')^3} + \frac{c_1}{(f')^2} = c_{11}, \tag{5.8}$$

and

$$\frac{g''}{(g')^3} + \frac{c_1}{(g')^2} = c_{12}, \tag{5.9}$$

where $c_{11}, c_{12} \in \mathbb{R}$ and $c_{12} = -c_1 - c_{11}$. From (5.8), we have

$$f'(u) = \pm\left(\frac{c_{11}}{c_1} + \frac{c_{13}}{c_1}e^{2\mu u}\right)^{\frac{-1}{2}},\ c_{13} \in \mathbb{R},\ c_{13} \neq 0. \tag{5.10}$$

If $c_{11}=0$ in (5.10), then we can derive $f(u)=\mp\left(\frac{c_{13}}{c_1}\right)^{\frac{-1}{2}}e^{-c_1u}$. Otherwise, we get

$$f(u)=-\frac{1}{\sqrt{c_1c_{11}}}\tanh^{-1}\left(\sqrt{1+\frac{c_{13}}{c_{11}}e^{2c_1u}}\right)=-\frac{1}{2\sqrt{c_1c_4}}\ln\left|\frac{1+\sqrt{1+\frac{c_{13}}{c_{11}}e^{2c_1u}}}{1-\sqrt{1+\frac{c_{13}}{c_{11}}e^{2c_1u}}}\right|.$$

Same solutions are also satisfied to (5.9). Up to suitable constants we complete the proof.

Theorem 5.3. *Let M^3 be a translation hypersurface of type 3 in $\mathbb{I}^4$ with nonzero constant mean curvature H_0. Then, up to suitable constants and translations of u,v,w, one of the following cases occurs:*

(i) $f(u)=\lambda u,\ g(v)=\left(\frac{-2\mu}{3H_0}v\right)^{\frac{1}{2}},\ h(w)=\mu w,\ \lambda\mu\neq 0;$

(ii) $f(u)=\lambda u^{\frac{1}{2}},\ g(v)=\mu v^{\frac{1}{2}},\ h(w)=\xi w,\ \lambda\mu\xi\neq 0;$

(iii) $f(u)=\lambda u,\ g(v)=\mu v,\ h(w)=\frac{3H_0(\lambda\mu)^2}{2\left[(\lambda\mu)^2+\lambda^2+\mu^2\right]}w^2,\ \lambda\mu\neq 0,$

where $\lambda,\mu,\xi\in\mathbb{R}$,

Proof. Due to $H_0\neq 0$, h cannot be constant in (5.2). We have to distinguish several cases to solve (5.2):

(1) $h'=h_0\in\mathbb{R},\ h_0\neq 0$. Then we write $h(w)=h_0w+c_1,\ c_1\in\mathbb{R}$. (5.2) reduces to

$$\frac{3H}{h_0}=\frac{f''}{(f')^3}+\frac{g''}{(g')^3},\tag{5.11}$$

where the roles of f and g are symmetric and so the situation on g is only considered.

(a) $g''=0,\ g(v)=c_2v+c_3,\ c_2,c_3\in\mathbb{R},\ c_2\neq 0$. Then (5.11) reduces to

$$\frac{f''}{(f')^3}=\frac{3H}{h_0}\tag{5.12}$$

and solving (5.12) gives

$$f(u)=\pm\frac{1}{\frac{3H_0}{h_0}}\sqrt{-\frac{6H_0}{h_0}u+c_4}+c_5,\ c_4,c_5\in\mathbb{R}.$$

Up to congruency of $\mathbb{I}^4$ one may assume $c_5=0$ and up to a translation on u, v and w choose $c_1=c_3=c_4=0$. Furthermore by putting $\lambda=\frac{\pm h_0\sqrt{-\frac{6H_0}{h_0}}}{3H_0}$, $c_2=\mu$ and $h_0=\xi$ we conclude that M^3 is congruent to the hypersurface given in the case (i) of the theorem.

(b) $g''\neq 0$. Hence (5.11) implies

$$\frac{f''}{(f')^3}=\frac{3H}{h_0}-c_6\text{ and }\frac{g''}{(g')^3}=c_6,\ c_6\in\mathbb{R},\ c_6\neq 0.\tag{5.13}$$

Solving (5.13) gives

$$f(u) = \pm \frac{1}{\frac{3H_0}{h_0} - c_6} \sqrt{-2\left(\frac{3H_0}{h_0} - c_6\right)u + c_7} + c_8$$

and

$$g(v) = \pm \frac{1}{c_6} \sqrt{-2c_6 v + c_9} + c_{10},$$

for $c_7, ..., c_{10} \in \mathbb{R}$. As in previous case, after applying suitable translations and choosing constants, the case (ii) of the theorem is proved.

(2) $h'' \neq 0$. If $f'' = 0 = g''$, then (5.2) leads to

$$h(w) = c_{11}w^2 + c_{12}w + c_{13},$$

where $c_{11}, c_{12}, c_{13} \in \mathbb{R}$. Up suitable translations and constants this implies the proof of the case (iii) of the theorem. If $f''g'' \neq 0$, dividing (5.2) with h' and taking its partial derivative with respect to w, we deduce

$$-3H_0 \frac{h''}{(h')^2} = \left(\frac{h''}{h'}\right)' \left[1 + \frac{1}{(f')^2} + \frac{1}{(g')^2}\right]. \tag{5.14}$$

Both-hand side must be nonzero in (5.14) and thus we can rewrite it as follows:

$$-3H_0 \frac{h''}{(h')^2} \left[\left(\frac{h''}{h'}\right)'\right]^{-1} = 1 + \frac{1}{(f')^2} + \frac{1}{(g')^2}. \tag{5.15}$$

This is a contradiction due to the fact that the right-hand side of (5.15) cannot be a constant. This completes the proof.

Theorem 5.3. immediately implies

Corollary 5.1. *Let M^3 be a translation hypersurface of type 3 in $\mathbb{I}^4$. Then, $H = const. \neq 0$ implies $K = 0$.*

6. Translation hypersurfaces of type 4

For a translation hypersurface of type 4, the matrices of the fundemantal forms are given by

$$[g_{ij}] = \begin{pmatrix} 2 + (f')^2 & 2 + f'g' & 5 + f'h' \\ 2 + f'g' & 2 + (g')^2 & 5 + g'h' \\ 5 + f'h' & 5 + g'h' & 37 + (h')^2 \end{pmatrix},$$

$$[g_{ij}]^{-1} = \frac{1}{49(f' - g')^2} \times$$

$$\times\begin{pmatrix} 37(g')^2+2(h')^2-10g'h'+49 & 5f'h'+5g'h'-37f'g'-2(h')^2-49 & 5f'g'+2g'h'-2f'h'-5(g')^2 \\ 5f'h'+5g'h'-37f'g'-2(h')^2-49 & 37(f')^2+2(h')^2-10f'h'+49 & 5f'g'+2f'h'-2g'h'-5(f')^2 \\ 5f'g'+2g'h'-2f'h'-5(g')^2 & 5f'g'+2f'h'-2g'h'-5(f')^2 & 2(f')^2+2(g')^2-4f'g' \end{pmatrix}$$

and

$$[h_{ij}] = \frac{2}{f'-g'}\begin{pmatrix} f'' & 0 & 0 \\ 0 & g'' & 0 \\ 0 & 0 & h'' \end{pmatrix}.$$

Hence the Gauss-Kronecker and the mean curvature are respectively

$$K = \frac{8f''g''h''}{49(f'-g')^5} \tag{6.1}$$

and

$$\begin{aligned} 3H = \tfrac{2}{49(f'-g')^3}\Big\{&\Big[37(g')^2+2(h')^2-10g'h'+49\Big]f''+ \\ +&\Big[37(f')^2+2(h')^2-10f'h'+49\Big]g''+2h''(f'-g')^2\Big\}. \end{aligned} \tag{6.2}$$

The roles of f and g are symmetric in (6.2) and, while solving it, the situations depending on f are only considered.

Theorem 6.1. *If a translation hypersurface of type 4 in $\mathbb{I}^4$ has nonzero constant Gauss-Kronecker curvature K_0, then it is a cylindrical hypersurface with non-isotropic rulings, namely $K_0 = 0$.*

Proof. Assume that $K = K_0 \neq 0$, it then follows from (6.1) that $f''g''h'' \neq 0$. Hence (6.1) reduces to

$$\frac{49K_0}{8h_0} = \frac{f''g''}{(f'-g')^5}, \tag{6.3}$$

where $h'' = h_0 \neq 0$, $h_0 \in \mathbb{R}$. The partial derivative of (6.3) with respect to u yields

$$f'''(f'-g') - 5(f'')^2 = 0. \tag{6.4}$$

The fact that the coefficient of the term g' in (6.4) must be zero leads to the contradiction $f'' = 0$. Therefore K vanishes and at least one of f, g, h is a linear function with respect to the given variable; that is, at least one of the generating curves turns to be a non-isotropic line. Without loss of generality we may assume that f is linear, i.e. $f(u) = c_1u + c_2$, $c_1, c_2 \in \mathbb{R}$. Hence, one can be parameterized by

$$\begin{aligned} &r(u,v,w) = u(c_1,1,1,1)+ \\ &\left(c_2+g(v)+h(w), v+6w, v-w, -v+w+\tfrac{5\pi}{6}\right). \end{aligned}$$

which means that it is congruent to a cylindrical hypersurface with non-isotropic rulings.

Theorem 6.2. *Let M^3 be a minimal translation hypersurface of type 4 in $\mathbb{I}^4$. Then M^3 is either a non-isotropic hyperplane or, up to suitable constants and translations of u, v, w, one of the following cases holds:*

(i) $f(u) = \lambda u$, $g(v) = \lambda v - \frac{1}{\mu}\ln|\mu v|$, $h(w) = \frac{5\lambda}{2}w + \frac{1}{\mu}\ln\left|\cos\frac{7\sqrt{2+\lambda^2}\mu}{2}w\right|$, $\mu \neq 0$;

(ii) $f(u) = \lambda u - \frac{1}{\mu} \ln|\cos \xi w|$, $g(v) = \lambda v + \frac{1}{\mu} \ln|\cos \xi w|$, $h(w) = \frac{37\lambda}{5} w$, $\mu\xi \neq 0$,

where $\lambda, \mu, \xi \in \mathbb{R}$,

Proof. The (6.2) follows

$$\begin{aligned} 0 &= \left[37 (g')^2 + 2 (h')^2 - 10 g' h' + 49\right] f'' \\ &+ \left[37 (f')^2 + 2 (h')^2 - 10 f' h' + 49\right] g'' + 2h'' (f' - g')^2 . \end{aligned} \tag{6.5}$$

We have two cases to solve (6.5):

(1) $f' = f_0$, $f_0 \in \mathbb{R}$. (6.5) can be rewritten as

$$\frac{g''}{(f_0 - g')^2} + \frac{2h''}{2(h')^2 - 10 f_0 h' + 37 f_0^2 + 49} = 0. \tag{6.6}$$

The situation that $g'' = h'' = 0$, $g' \neq f_0$, leads M^3 to be a non-isotropic hyperplane. If $g''h'' \neq 0$, (6.6) implies

$$\frac{g''}{(f_0 - g')^2} = c_1 = \frac{-2h''}{2(h')^2 - 10 f_0 h' + 37 f_0^2 + 49}, \tag{6.7}$$

where $c_1 \in \mathbb{R}$, $c_1 \neq 0$. After solving (6.7), we conclude

$$g(v) = f_0 v - \frac{1}{c_1} \ln|c_1 v + c_2| + c_3$$

and

$$h(w) = \frac{5 f_0}{2} w + \frac{1}{c_1} \ln\left|\cos\left(-\frac{7 c_1 \sqrt{2 + f_0^2}}{2} w + c_4\right)\right| + c_5,$$

where $c_2, ..., c_5 \in \mathbb{R}$. Up to congruency of $\mathbb{I}^4$ one may assume $c_3 = c_5 = 0$ and up to a translation on v and w, choose $c_2 = c_4 = 0$. Furthermore by putting $\lambda = f_0$ and $\mu = c_1$, we obtain that M^3 is congruent to the hypersurface given in the case (i) of the theorem.

(2) $f'' \neq 0$. By symmetry, we deduce $g'' \neq 0$. We have two cases:

(a) $h' = h_0$, $h_0 \in \mathbb{R}$. (6.5) can be rewritten as

$$\frac{f''}{37 (f')^2 - 10 h_0 f' + 49 + 2h_0^2} = c_6 = \frac{-g''}{37 (g')^2 - 10 h_0 g' + 49 + 2h_0^2}, \tag{6.8}$$

for $c_6 \in \mathbb{R}$, $c_6 \neq 0$. Solving (6.8), we conclude

$$f(u) = \frac{-1}{37 c_6} \ln|\cos(c_6 k u + c_7)| + \frac{5 h_0}{37} u + c_8$$

and

$$g(v) = \frac{1}{37 c_6} \ln|\cos(-k\lambda v + c_9)| + \frac{5 h_0}{37} v + c_{10},$$

where $c_7, ..., c_{10} \in \mathbb{R}$ and $k = \sqrt{1813 + 12 h_0^2}$. As in previous case; after applying suitable translations and choosing constants, we complete the proof of the case (ii) of the theorem

(b) $h'' \neq 0$. Taking partial derivative of (6.5) with respect to u, v, w and then dividing $f''g''h''$ yields

$$5\frac{f'''}{f''} + 5\frac{g'''}{g''} + 2\frac{h'''}{h''} = 0,$$

which leads to

$$f''' = c_{11}f'', \ g''' = c_{12}g'', \ h''' = c_{13}h'', \tag{6.9}$$

for $c_{11}, c_{12}, c_{13} \in \mathbb{R}$ with $5c_{11} + 5c_{12} + 2c_{13} = 0$. Integrating (6.9) gives

$$f'' = c_{11}f' + c_{14}, \ g'' = c_{12}g' + c_{15}, \ h'' = c_{13}h' + c_{16},$$

for $c_{14}, c_{15}, c_{16} \in \mathbb{R}$. On the other hand, taking partial derivative of (6.5) with respect to w and dividing h'' leads to

$$(4h' - 10g')\, f'' + (4h' - 10f')\, g'' + 2h'''\, (f' - g')^2 = 0. \tag{6.10}$$

If we substitute (6.9) into (6.10), then

$$(4h' - 10g')\,(c_{11}f' + c_{14}) + (4h' - 10f')\,(c_{12}g' + c_{15}) + 2c_{13}\,(f' - g')^2 = 0,$$

which is a polynomial equation on f' or g'. This immediately gives $c_{13} = 0$, i.e.

$$(4h' - 10g')\,(c_{11}f' + c_{14}) + (4h' - 10f')\,(c_{12}g' + c_{15}) = 0. \tag{6.11}$$

Taking partial derivative of (6.11) with respect to w and dividing $4h''$ implies

$$c_{11}f' + c_{12}g' + c_{14} + c_{15} = 0,$$

which means $c_{11} = c_{12} = 0$, i.e. $f''' = g''' = h''' = 0$. Considering it into (6.5) and then taking parital derivatives its with respect to u and v yield

$$-4f''g''h'' = 0,$$

which gives a contradiction.

Theorem 6.3. *Let M^3 be a translation hypersurface of type 4 in $\mathbb{I}^4$ with nonzero constant mean curvature H_0. Then, up to suitable constants and translations of u, v, w, one of the following cases holds:*

(i) $f(u) = \lambda u$, $g(v) = \mu v$, $h(w) = \frac{147H_0}{8}(\lambda - \mu)\, w^2$, $\lambda \neq \mu$;
(ii) $f(u) = \lambda u$, $g(v) = \lambda v + \mu v^{\frac{1}{2}}$, $h(w) = \xi w$, $\mu \neq 0$,

where $\lambda, \mu, \xi \in \mathbb{R}$.

Proof. We have several cases to solve (6.2):

(1) $f' = f_0 \in \mathbb{R}$. (6.2) then reduces to

$$c_1 \left(f_0 - g'\right)^3 = \left[2 \left(h'\right)^2 - 10 f_0 h' + 37 f_0^2 + 49\right] g'' + 2h'' \left(f_0 - g'\right)^2, \tag{6.12}$$

where $c_1 = \frac{147 H_0}{2} \neq 0$. If $g' = g_0 \in \mathbb{R}$ and $f_0 \neq g_0$ in (6.12), then we immediately have

$$h\left(w\right) = \frac{147 H_0}{8} \left(f_0 - g_0\right) w^2 + c_2 w + c_3,\ \ c_2, c_3 \in \mathbb{R}.$$

If we put $\lambda = f_0$, $\mu = g_0$ and apply suitable translations on u, v, w, then we prove that M^3 is congruent to the hypersurface given in the case (i) of the theorem. Next we assume $g'' \neq 0$ and consider the following cases:

(a) $h' = h_0 \in \mathbb{R}$. (6.12) follows

$$\frac{g''}{\left(f_0 - g'\right)^3} = c_4, \tag{6.13}$$

for $c_4 = \frac{c_1}{37 f_0^2 + 2h_0^2 - 10 h_0 f_0 + 49}$. Solving (6.13) leads to

$$g\left(v\right) = f_0 v \pm \frac{1}{c_4} \left(2 c_4 v + c_5\right)^{\frac{1}{2}} + c_6,\ \ c_5, c_6 \in \mathbb{R}.$$

As in previous case; after applying suitable translations and choosing constants, we prove the case (ii) of the theorem.

(b) $h'' \neq 0$. The partial derivative of (6.12) with respect to w gives

$$\frac{g''}{\left(f_0 - g'\right)^2} + \frac{h'''}{h'' \left(2h' - 5 f_0\right)} = 0, \tag{6.14}$$

where $h''' \neq 0$ due to $g'' \neq 0$. (6.14) implies

$$\frac{g''}{\left(f_0 - g'\right)^2} = c_7 = -\frac{h'''}{h'' \left(2h' - 5 f_0\right)}, \tag{6.15}$$

where $c_7 \in \mathbb{R}$, $c_7 \neq 0$. Considering (6.15) into (6.12) leads to

$$\lambda \left(f_0 - g'\right) = \left[2 \left(h'\right)^2 - 10 h' f_0 + 37 f_0^2 + 49\right] c_7 + 2h''. \tag{6.16}$$

The contradiction $g'' = 0$ is obtained by taking partial derivative of (6.16) with respect to v.

(2) $f'' \neq 0$. The symmetry follows $g'' \neq 0$. We have two cases:

(a) $h'' = 0$, $h' = h_0$. (6.2) can be rewritten as

$$\begin{aligned} &\tfrac{147 H_0 \left(f' - g'\right)^3}{2} = \\ &\left[37 \left(g'\right)^2 + 2h_0^2 - 10 g' h_0 + 49\right] f'' + \left[37 \left(f'\right)^2 + 2h_0^2 - 10 f' h_0 + 49\right] g''. \end{aligned} \tag{6.17}$$

Twice partial derivatives of (6.17) with respect to u and v give

$$\left(\frac{f'''}{f'}\right)' g'' + \left(\frac{g'''}{g''}\right)' f'' = 0,$$

which yields

$$\left(\frac{f'''}{f''}\right)' = c_8 f'', \quad \left(\frac{g'''}{g''}\right)' = -c_8 g'',\ \ c_8 \in \mathbb{R}. \tag{6.18}$$

Integrating (6.18) leads to

$$\frac{f'''}{f''} = c_8 f' + c_9, \quad \frac{g'''}{g''} = -c_8 g' + c_{10}, \ c_9, c_{10} \in \mathbb{R}.$$

Now taking partial derivative of (6.17) with respect to u and dividing f'' gives

$$\frac{441 H_0 \left(f' - g'\right)^2}{2} = \left[37 \left(g'\right)^2 + 2h_0^2 - 10 g' h_0 + 49\right] \left(c_8 f' + c_{10}\right) + \left[74 f' + 10 h_0\right] g''.$$

The last equation is a polynomial equation on f'; however, the leading coefficient is $\frac{441 H_0}{2}$ which cannot be zero. This is a contradiction.

(b) $h'' \neq 0$. Let us put $\Phi = \frac{147 H_0}{2} \left(f' - g'\right)^3$. Then considering (6.2) and the equation $\frac{\Phi_{uvw}}{f'' g'' h''} = 0$ gives

$$5 \frac{f'''}{f''} + 5 \frac{g'''}{g''} + 2 \frac{h'''}{h''} = 0,$$

or

$$f''' = c_{11} f'', \ g''' = c_{12} g'', \ h''' = c_{13} h'', \tag{6.19}$$

for $c_{11}, c_{12}, c_{13} \in \mathbb{R}$ with $5c_{11} + 5c_{12} + 2c_{13} = 0$. Integrating (6.19) gives

$$f'' = c_1 f' + c_{14}, \ g'' = c_2 g' + c_{15}, \ h'' = c_3 h' + c_{16}. \tag{6.20}$$

Plugging (6.20) into (6.2) gives a polynomial equation on f'; however, the leading coefficient is $\frac{147 H_0}{2}$ which cannot be zero. This completes the proof.

Theorem 6.3. immediately implies

Corollary 6.1. *Let M^3 be a translation hypersurface of type 4 in $\mathbb{I}^4$. Then, $H = const. \neq 0$ implies $K = 0$.*

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ON RICCI TENSOR IN THE GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract. The object of the present paper is to study the properties of generalized Sasakian-space-forms. We prove the results related to Ricci symmetric, Ricci recurrent, cyclic parallel and Codazzi type Ricci tensors. Results on Ricci soliton and gradient Ricci soliton are proved. Also, we provide the examples of generalized Sasakian-space-forms which are verified our results.

1. Introduction

An almost Hermitian manifold endowed with an almost complex structure J is said to be a generalized complex-space-form if the curvature tensor R is non-vanishing and satisfies

$$R(X,Y)Z = F_1\{g(Y,Z)X - g(X,Z)Y\} + F_2\{g(X,JZ)JY$$
$$-g(Y,JZ)JX + 2g(X,JY)JZ\},$$

for smooth functions F_1, F_2 and all the vector fields X, Y, Z. Motivated by this fact, P. Alegre et al. [1] defined the generalized Sasakian-space-forms and proved many new results. They also validate the existence of such space forms by providing non-trivial examples.

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An almost contact metric manifold M equipped with almost contact structure (ϕ, ξ, η, g) is said to be a generalized Sasakian-space-form if its non-vanishing curvature tensor R satisfies the relation

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ &\quad -g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi\}, \end{aligned} \tag{1.1}$$

for smooth functions f_1, f_2, f_3 on M and all the vector fields X, Y, $Z \in T(M)$, where $T(M)$ denotes the tangent bundle of the manifold M. We will denote the generalized Sasakian-space-form by $M(f_1, f_2, f_3)$. A generalized Sasakian space form can be cosymplectic, Sasakian and Kenmotsu space forms if M is cosymplectic with $f_1 = \frac{c}{4} = f_2 = f_3$, M is Sasakian and $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$ and M is Kenmotsu together with $f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4}$ respectively, where c is constant. Thus we can say that the generalized Sasakian-space-forms are the natural generalization of cosymplectic, Sasakian and Kenmotsu space forms. Various new results of the generalized Sasakian-space-forms have been noticed in ([1]-[4], [14]-[16], [21]-[23], [28]).

In the beginning of $80's$, Hamilton [18] introduced the notion of Ricci flow to obtain a canonical metric on a differentiable manifold. Since then it became a powerful tool to study Riemannian manifolds of positive curvature. To prove the Poincaré conjecture, Perelman ([25], [26]) used Ricci flow and its surgery. Also Brendle and Schoen [8] proved the differentiable sphere theorem by using Ricci flow. The evolution equation for metrics on a Riemannian manifold, called Ricci flow and defined as

$$\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij}, \quad g(0) = g_0,$$

for g_0 fixed metric on M, where S_{ij} denotes the components of Ricci tensor. The solutions of Ricci flow are called the Ricci solitons if they are governed by a one parameter family of diffeomorphisms and scalings. A triplet (g, V, λ) on a Riemannian manifold (M, g) is called a Ricci soliton [19], natural generalized of Einstein metric, and satisfies

$$\frac{1}{2} L_V g + S + \lambda g = 0, \tag{1.2}$$

where S is the Ricci tensor, $L_V g$ denotes the Lie derivative of Riemannian metric g along the vector field V on M and λ is a real constant [19]. A Ricci soliton is said to be steady, expanding or shrinking if $\lambda = 0, \lambda > 0$ or $\lambda < 0$, respectively. Ricci solitons are self similar

solution of the Ricci flow, possible singularity models of the Ricci flow and critical points of Perelman's λ-entropy and μ-entropy [9]. Many authors studied the properties of of the Ricci solitons but few are ([9]-[13], [22], [27]). The metric g is said to be gradient Ricci soliton if the vector field V is the gradient of a potential function $-f$. In such case (1.2) assumes the form

$$\nabla\nabla f = S + \lambda g, \tag{1.3}$$

where ∇ represents the Levi-Civita connection of the metric g.

Motivated by above studies, present authors continue the study of generalized Sasakian-space-forms and Ricci solitons. We organize the paper as: After introduction in section 2, we brief the basic results of contact metric manifolds and generalized Sasakian-space-forms. In section 3, we present the equivalent conditions for scalar curvature, necessary and sufficient condition for Ricci symmetric and cyclic parallel Ricci tensor. We also prove that the generalized Sasakian-space-forms are certain class of almost contact metric manifolds under certain restrictions. The properties of Ricci and gradient Ricci solitons are given in section 4. Section 5 deals with examples of the generalized Sasakian-space-forms which are verified our results.

2. Preliminaries

Let a differentiable manifold M $(dimM = 2n+1)$ of differentiability class C^∞ carries a global differentiable 1-form η $(\eta \wedge (d\eta)^n \neq 0)$, a global non-vanishing vector field or the characteristic vector field ξ and the structure vector field ϕ, then M is said to have a (ϕ, ξ, η)-structure or almost contact structure (ϕ, ξ, η) to M if

$$\eta(\xi) = 1 \ and \ \phi^2 = -I + \eta \otimes \xi, \tag{2.4}$$

where I denotes the identity transformation [6]. From (2.4), it can be easily see that $\phi\xi = 0$, $\eta.\phi = 0$ and rank $\phi = 2n$. A Riemannian metric g of type $(0, 2)$ is said to be compatible with the almost contact structure (ϕ, ξ, η) if the relations

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X) \tag{2.5}$$

hold for arbitrary vector fields X and Y on M. An almost contact structure (ϕ, ξ, η) equipped with a compatible Riemannian metric g is known as almost contact metric structure (ϕ, ξ, η, g) and the manifold M endowed with the almost contact metric structure is called an almost contact metric manifold. If the fundamental 2-form of M defined as $\Phi(X, Y) = g(X, \phi Y)$ for

arbitrary vector fields X and Y on M and satisfies $d\eta = \Phi$, then an almost contact metric manifold reduces to a contact metric manifold. A normal contact metric manifold is an almost contact metric manifold with $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ represents the Nijenhuis tensor of ϕ and d is an exterior derivative. A normal contact metric manifold is Sasakian manifold. A Sasakian manifold is always a K-contact manifold (ξ is Killing) although in dimension 3, K-contact is Sasakian. It is noticed that the generalized Sasakian-space-forms $M(f_1, f_2, f_3)$ satisfy the followings:

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.6}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.7}$$

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.8}$$

$$R(\xi, X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.9}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.10}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3 \tag{2.11}$$

for all X, Y, $Z \in T(M)$. Here Q denotes the Ricci operator such that $S(X,Y) = g(QX,Y)$ and r is the scalar curvature to M.

Before going to prove our main results in next sections, we recall the followings:

Definition 2.1. *A Riemannian manifold M of dimension n is said to be Ricci symmetric if the non-vanishing Ricci tensor S of M satisfies $(\nabla_X S)(Y,Z) = 0 \ \forall\ X, Y, Z \in T(M)$.*

Definition 2.2. *An n-dimensional Riemannian manifold M endowed with the non-zero Ricci tensor S is said to be a Ricci recurrent* [24] *if $(\nabla_X S)(Y,Z) = A(X)S(Y,Z)$ holds for all X, Y, $Z \in T(M)$. Here A is a non-zero 1-form.*

Definition 2.3. *A non-zero Ricci tensor S of an n-dimensional Riemannian manifold M is said to be Codazzi type* [5]*, or cyclic parallel* [17] *if S satisfies $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$, or $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$, respectively for all X, Y, $Z \in T(M)$.*

3. Main Results

In this section, we study the properties of Ricci symmetric, Ricci recurrent, Codazzi type Ricci tensor and cyclic parallel Ricci tensor on a generalized Sasakian-space-form.

We recall the following theorem of P. Alegre et al. [1] that we will be useful to prove our main results.

Theorem 3.1. *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-spce-form. Let M is a contact metric manifold, then $f_1 - f_3$ is constant on M (see Theorem 3.10, page 164,* [1]*).*

Lemma 3.1. *On a generalized Sasakian-space-form $M(f_1, f_2, f_3)$, the following conditions are equivalent:*
(i) scalar curvature of $M(f_1, f_2, f_3)$ is constant,
(ii) $(2n-1)f_1 + 3f_2$ is constant,
(iii) $3f_2 + (2n-1)f_3$ is constant.

Proof. Let us suppose that the scalar curvature of $M(f_1, f_2, f_3)$ is constant and therefore $dr(X) = 0$ for arbitrary vector field X on $M(f_1, f_2, f_3)$. From (2.11), we have

$$r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\} = 2n\{(2n-1)f_1 + 3f_2 + 2(f_1 - f_3)\}.$$

In view of Theorem 3.1 and above discussion, we have $d((2n-1)f_1 + 3f_2)(X) = 0$, for the vector field X on $M(f_1, f_2, f_3)$. This shows that $(2n-1)f_1 + 3f_2$ is constant on $M(f_1, f_2, f_3)$. Hence $(i) \Rightarrow (ii)$. Next,

$$(2n-1)f_1 + 3f_2 = (2n-1)(f_1 - f_3) + 3f_2 + (2n-1)f_3,$$

which shows that

$$d((2n-1)f_1 + 3f_2)(X) = d(3f_2 + (2n-1)f_3)(X).$$

If $(2n-1)f_1 + 3f_2$ is constant on $M(f_1, f_2, f_3)$, then $3f_2 + (2n-1)f_3$ is also constant on it. Now we have to prove that $(iii) \Rightarrow (i)$. Equation (2.11) can be written as

$$r - 2n(2n+1)(f_1 - f_3) = 2n\{3f_2 + (2n-1)f_3\}.$$

It is obvious from the above equation and Theorem 3.1 that $dr(X) = 2n\, d(3f_2 + (2n-1)f_3)(X)$. This informs that if $3f_2 + (2n-1)f_3$ is constant on $M(f_1, f_2, f_3)$, then the scalar curvature of $M(f_1, f_2, f_3)$ will be also constant. This complete the proof.

In [28], authors proved that on a Ricci symmetric generalized Sasakian-space-form, $f_1 - f_3$ is constant. Motivated by this, we are going to prove the following:

Theorem 3.2. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. Then $M(f_1, f_2, f_3)$ is Ricci symmetric if and only if either the characteristic vector field of M is parallel and scalar curvature is constant or $3f_2 + (2n-1)f_3 = 0$.*

Proof. Equation (2.6) can be rewritten as

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z), \tag{3.12}$$

where $a = 2nf_1 + 3f_2 - f_3$ and $b = -3f_2 - (2n-1)f_3$ are smooth functions on $M(f_1, f_2, f_3)$. In consequence of (3.12) and the Theorem 3.1, we have

$$da(X) + db(X) = 0, \tag{3.13}$$

for arbitrary vector field $X \in T(M)$. Covariant derivative of (3.12) along the vector field X gives

$$\begin{aligned}(\nabla_X S)(Y, Z) &= da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) \\ &\quad + b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)].\end{aligned} \tag{3.14}$$

Setting $Z = \xi$ and using (2.4) and (3.13) in (3.14), we find

$$(\nabla_X S)(Y, \xi) = b[(\nabla_X \eta)(Y) + (\nabla_X \eta)(\xi)\eta(Y)].$$

Since $g(\xi, \xi) = 1 \Rightarrow g(\nabla_X \xi, \xi) = 0$, therefore above equation takes the form

$$(\nabla_X S)(Y, \xi) = b(\nabla_X \eta)(Y). \tag{3.15}$$

Let us suppose that $M(f_1, f_2, f_3)$ is Ricci symmetric, that is $\nabla S = 0$, and therefore (3.14) assumes the form

$$da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) + b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)] = 0. \tag{3.16}$$

Changing Z by ξ in (3.16) and then using (2.4), (2.5) and (3.13), we have

$$b(\nabla_X \eta)(Y) = 0. \tag{3.17}$$

This reflects that either $b = 0$, *i.e.*, $3f_2 + (2n-1)f_3 = 0$ and $(\nabla_X \eta)(Y) \neq 0$ or $(\nabla_X \eta)(Y) = 0 \Rightarrow \nabla_X \xi = 0$, that is the characteristic vector filed of the manifold is parallel and $b \neq 0$. Thus in view of $\nabla_X \xi = 0$, equations (2.5), (3.13) and (3.16) reflect that

$$da(X)g(\phi Y, \phi Z) = 0, \qquad \forall\ X, Y, Z \in T(M). \tag{3.18}$$

In general, $g(\phi Y, \phi Z) \neq 0$ on almost contact metric manifold and thus $a = constant \Rightarrow r = constant$, where Theorem 3.1 and Lemma 3.1 are used. To prove the converse part first we suppose that $3f_2 + (2n-1)f_3 = 0$ and $(\nabla_X \eta)(Y) \neq 0$ and thus with (3.13) and (3.14), we find that $\nabla S = 0$. Secondly, we consider that $3f_2 + (2n-1)f_3 \neq 0$ and $(\nabla_X \eta)(Y) = 0$ with constant scalar curvature on $M(f_1, f_2, f_3)$. Since r is constant, therefore by Lemma 3.1 it can easily verify that $a, b = constants$. Hence equation (3.14) shows that $\nabla S = 0$. Thus the Theorem is proved.

Theorem 3.3. [7] *Let M^{2n+1} be a contact metric manifold of dimension $(2n+1)$ and $R(X,Y)\xi = 0$ for all vector fields X and $Y \in T(M)$. Then M^{2n+1} is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature* 4.

By considering above discussions and Theorems 3.2 and 3.3, we state the following:

Corollary 3.1. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form and $3f_2 + (2n-1)f_3 \neq 0$. Then M is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

Next we are going to study the Ricci recurrent generalized Sasakian-space-forms and prove its existence. We suppose that the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is Ricci recurrent, that is the non-vanishing Ricci tensor S of $M(f_1, f_2, f_3)$ satisfies

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z), \tag{3.19}$$

for arbitrary vector fields X, Y and Z on $M(f_1, f_2, f_3)$, where A is any non-zero 1-form [24]. Setting $Z = \xi$ and using (2.6) in (3.19), we get

$$(\nabla_X S)(Y, \xi) = 2n(f_1 - f_3)A(X)\eta(Y), \tag{3.20}$$

i.e.

$$b(\nabla_X \eta)(Y) = 2n(f_1 - f_3)A(X)\eta(Y), \tag{3.21}$$

where equations (3.15) and (3.20) are used. Putting $Y = \xi$ in (3.21) and then use of (2.4) and (2.5) we have $A = 0$, provided $f_1 \neq f_3$. Hence we observe the following:

Theorem 3.4. *There does not exist a Ricci recurrent generalized Sasakian-space-form $M(f_1, f_2, f_3)$, provided $f_1 \neq f_3$.*

Theorem 3.5. *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form and $3f_2 + (2n-1)f_3$ is a non-zero constant on it. Then the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel if and only if the characteristic vector field of the manifold is Killing.*

Proof. In (3.14) setting $Y = Z = e_i$, where $\{e_i, i = 1, 2, ..., 2n+1\}$ be a set of orthonormal vector field of the tangent space at each point of the manifold M and taking summation over i $(1 \leq i \leq 2n+1)$, we find that

$$dr(X) = (2n+1)da(X) + db(X).$$

Since $3f_2 + (2n-1)f_3$ is a non-zero constant and therefore by Lemma 3.1 it is obvious that the scalar curvature of $M(f_1, f_2, f_3)$ is constant. Thus above equation gives

$$(2n+1)da(X) + db(X) = 0. \tag{3.22}$$

From (3.14), we have

$$\begin{aligned}
&(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\
&= da(X)g(Y, Z) + da(Y)g(Z, X) + da(Z)g(X, Y) \\
&\quad + db(X)\eta(Y)\eta(Z) + db(Y)\eta(Z)\eta(X) + db(Z)\eta(X)\eta(Y) \\
&\quad + b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) + (\nabla_Y \eta)(X)\eta(Z) \\
&\quad + (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Z \eta)(Y)\eta(X) + (\nabla_Z \eta)(X)\eta(Y)].
\end{aligned} \tag{3.23}$$

Let us suppose that the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel. Then (3.23) converts into the form

$$\begin{aligned}
&da(X)g(Y, Z) + da(Y)g(Z, X) + da(Z)g(X, Y) \\
&\quad + db(X)\eta(Y)\eta(Z) + db(Y)\eta(Z)\eta(X) + db(Z)\eta(X)\eta(Y) \\
&\quad + b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) + (\nabla_Y \eta)(X)\eta(Z) \\
&\quad + (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Z \eta)(Y)\eta(X) + (\nabla_Z \eta)(X)\eta(Y)] = 0.
\end{aligned} \tag{3.24}$$

Changing Y and Z by ξ and using (2.4) and (2.5) in (3.24), we get

$$da(X) + db(X) + 2\{da(\xi) + db(\xi)\}\eta(X) + 2(\nabla_\xi \eta)(X) = 0.$$

In consequence of (3.13), last expression becomes

$$(\nabla_\xi \eta)(X) = 0. \tag{3.25}$$

Replacing Y and Z with e_i in (3.24) and then taking summation for i, $1 \leq i \leq (2n+1)$, we have

$$(n+1)\,da(X) + db(\xi)\eta(X) + b\sum_{i=1}^{2n+1}(\nabla_{e_i}\eta)(e_i)\eta(X) = 0, \tag{3.26}$$

where equations (2.4), (2.5), (3.13) and (3.25) are used. Putting $X = \xi$ and using (2.5) and (3.13) in (3.26), we obtain

$$n\,da(\xi) + b\sum_{i=1}^{2n+1}(\nabla_{e_i}\eta)(e_i) = 0, \tag{3.27}$$

Equations (3.13), (3.26) and (3.27) give

$$da(X) = da(\xi)\eta(X) = -db(X). \tag{3.28}$$

In consequence of (2.5), (3.13) and (3.24), we have

$$\begin{aligned} &da(X)g(\phi Y,\phi Z) + da(Y)g(\phi Z,\phi X) + da(Z)g(\phi X,\phi Y) \\ &\quad + b[(\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y) + (\nabla_Y\eta)(X)\eta(Z) \\ &\quad + (\nabla_Y\eta)(Z)\eta(X) + (\nabla_Z\eta)(Y)\eta(X) + (\nabla_Z\eta)(X)\eta(Y)] = 0. \end{aligned} \tag{3.29}$$

Setting $Z = \xi$ in (3.29), we get

$$(\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) = 0, \tag{3.30}$$

where equations (2.5), (3.13), (3.22), (3.25) and (3.28) are used. This shows that the characteristic vector field ξ of $M(f_1, f_2, f_3)$ is Killing. Conversely, we suppose that the equation (3.30) holds and $3f_2 + (2n-1)f_3$ is a non-zero constant on $M(f_1, f_2, f_3)$. With the help of (3.13), (3.22), (3.23), (3.30) and Lemma 3.1, we can prove that $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$. Hence the statement of the Theorem is satisfied.

Theorem 3.6. *If the Ricci tensor of $M(f_1, f_2, f_3)$ is of Codazzi type, then $M(f_1, f_2, f_3)$ is either a certain class of almost contact metric manifold whose characteristic vector field ξ satisfies (3.36) or Ricci symmetric.*

Proof. From (3.14), we have

$$\begin{aligned} (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) &= da(X)g(Y,Z) + db(X)\eta(Y)\eta(Z) \\ &\quad - da(Y)g(X,Z) - db(Y)\eta(X)\eta(Z) + b[(\nabla_X\eta)(Y)\eta(Z) \\ &\quad + (\nabla_X\eta)(Z)\eta(Y) - (\nabla_Y\eta)(X)\eta(Z) - (\nabla_Y\eta)(Z)\eta(X)]. \end{aligned} \tag{3.31}$$

Let us suppose that the Ricci tensor of $M(f_1, f_2, f_3)$ to be Codazzi type, that is $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$. Thus from equation (3.31), we have

$$0 = b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) - (\nabla_Y \eta)(X)\eta(Z) - (\nabla_Y \eta)(Z)\eta(X)]$$
$$+da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) - da(Y)g(X, Z) - db(Y)\eta(X)\eta(Z). \tag{3.32}$$

Changing Y and Z with ξ in (3.32), we obtain

$$b(\nabla_\xi \eta)(X) = 0, \tag{3.33}$$

where equations (2.4), (2.5) and (3.13) are used. Again setting $Z = \xi$ in (3.32) and then using equations (2.4), (2.5), (3.13) and (3.33), we conclude that

$$b[(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)] = 0. \tag{3.34}$$

This shows that either $b = 0 \iff 3f_2 + (2n-1)f_3 = 0$ or $(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0$, that is the contact 1-form η is closed. Now we have two cases:

Case I: Let us suppose that b is a non vanishing smooth function on $M(f_1, f_2, f_3)$ and $(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0$, that is the contact 1-form η is closed. Thus we have

$$g(\nabla_X \xi, Y) = g(X, \nabla_Y \xi).$$

Again putting $Y = \xi$ in (3.32), we find that

$$b(\nabla_X \eta)(Z) - da(\xi)\{g(X, Z) - \eta(X)\eta(Z)\} = 0, \tag{3.35}$$

where equations (2.4), (2.5), (3.13) and (3.33) are used. The straight forward calculation from (3.35) shows that

$$\nabla_X \xi = \nu\{X - \eta(X)\xi\}, \tag{3.36}$$

$\nu = \frac{da(\xi)}{b} \neq 0$. Equation (3.36) reveals that $M(f_1, f_2, f_3)$ under consideration is a certain class of almost contact metric manifold. If b is a non-zero constant, then $da(\xi) = 0 \iff \nabla_X \xi = 0$. Thus the characteristic vector field ξ of M is parallel. On the other hand if $\nu \in \mathfrak{R}$ ($\mathfrak{R}$ is a real number and $\nu \neq 0$), the equation (3.36) reflects that $M(f_1, f_2, f_3)$ with our assumption becomes ν-Kenmotsu manifold [20].

Case II: In this case we consider that $(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) \neq 0$ and $b = 0 \iff 3f_2 + (2n-1)f_3 = 0$ on $M(f_1, f_2, f_3)$. By considering this fact, (2.6) takes the form

$$S(Y, Z) = 2n(f_1 - f_3)g(Y, Z), \tag{3.37}$$

provided $f_1 \neq f_3$. Taking covariant derivative of (3.37) along the vector field X, we find that

$$(\nabla_X S)(Y,Z) = 0 \tag{3.38}$$

by virtue of Theorem 3.1. This tell us that the Ricci tensor of $M(f_1, f_2, f_3)$ is Ricci symmetric and manifold under consideration to be Einstein manifold. Hence the statement of the Theorem is satisfied.

4. Ricci soliton in generalized Sasakian-space-forms

This section deals with the study of Ricci soliton and gradient Ricci soliton in generalized Sasakian-space-forms $M(f_1, f_2, f_3)$. In [16], authors proved that on the generalized Sasakian-space-form a second order parallel symmetric tensor is proportional to a metric tensor g. Recently P. Majhi with U. C. De [22] studied the properties of Ricci soliton and gradient Ricci soliton in three dimensional generalized Sasakian-space-forms under certain restrictions.

Since $\nabla g = 0$ on M and therefore for a constant $\lambda \in \mathfrak{R}$ ($\mathfrak{R}$ being real number), $\nabla 2\lambda g = 0$ holds on M. Therefore equation (1.2) shows that $\mathcal{L}_V g + 2S$ is parallel. This discussion with Theorem 3.1 [for more details see p.4 , [16]] reflects that $\mathcal{L}_V g + 2S$ is a constant multiple of metric tensor g, that is $\mathcal{L}_V g + 2S = \alpha g$, where α is a constant. Thus $\mathcal{L}_V g + 2S + 2\lambda g = (\alpha + 2\lambda)g = 0 \Rightarrow \lambda = -\frac{\alpha}{2}$. We state the following lemma:

Lemma 4.1. *Let $M(f_1, f_2, f_3)$ is a $(2n+1)$-dimensional generalized Sasakian-space-form. A Ricci soliton (g, V, λ) on $M(f_1, f_2, f_3)$ to be shrinking and expanding if α is > 0 and < 0, respectively.*

In particular, if we suppose that $V = \xi$ on $M(f_1, f_2, f_3)$, then $(\mathcal{L}_\xi)g(\xi, \xi) = 0$. Setting $X = Y = V = \xi$ in (1.2) and then applying (2.4) and (2.10), we find that $\lambda = -2n(f_1 - f_3)$. Hence we can say the following:

Lemma 4.2. *A Ricci soliton (g, ξ, λ) in $M(f_1, f_2, f_3)$ ($dimM = 2n+1$) is shrinking, expanding and steady if $f_1 > f_3, f_1 < f_3$ and $f_1 = f_3$, respectively.*

Remark 4.1. *It is observed in Lemma 4.2 that the classification of Ricci flow is independent of smooth function f_2.*

Also we consider that V is a point wise collinear with ξ, that is $V = \beta\xi$, where β is a non-zero smooth function on $M(f_1, f_2, f_3)$. Thus we have

$$2S(X,Y) = -2\lambda g(X,Y) - (X\beta)\eta(Y) - (Y\beta)\eta(X) - \beta\{g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)\}. \tag{4.39}$$

Changing X and Y with ξ in (4.39) and then utilizing equations (2.4) and (2.5), we have

$$\lambda = -\{\xi\beta + 2n(f_1 - f_3)\}. \tag{4.40}$$

Again putting $X = \xi$ in (4.39) and making use of equations (2.4), (2.5), (2.10) and (4.40), we conclude

$$d\beta = -(\xi\beta)\eta, \tag{4.41}$$

which shows that β is constant and therefore from (4.40) $\lambda = -2n(f_1 - f_3)$ on $M(f_1, f_2, f_3)$. Let us suppose that the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel and therefore by the Theorem 3.5 we can say that the characteristic vector field of $M(f_1, f_2, f_3)$ is Killing. By considering this fact and (4.41), equation (4.39) takes the form

$$S(X, Y) = -2n(f_1 - f_3)g(X, Y), \tag{4.42}$$

for all $X, Y \in T(M)$. Equation (4.42) with Theorem 3.1 reveals that $M(f_1, f_2, f_3)$ is an Einstein manifold. It is obvious from (4.42) and Theorem 3.1 that the scalar curvature of $M(f_1, f_2, f_3)$ is constant. Hence we state the following:

Theorem 4.1. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form whose Ricci tensor is cyclic parallel. If the metric g is a Ricci soliton and V pointwise collinear with the characteristic vector field ξ on $M(f_1, f_2, f_3)$, then the scalar curvature to be constant and $M(f_1, f_2, f_3)$ is an Einstein manifold.*

Next we are going to study the properties of gradient Ricci soliton on generalized Sasakian-space-forms. Let the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel and g is a gradient Ricci soliton on $M(f_1, f_2, f_3)$. Thus we have from (1.3)

$$\nabla_Y Df = QY + \lambda Y, \tag{4.43}$$

for arbitrary vector field Y on $M(f_1, f_2, f_3)$, where D denotes the gradient operator of g. In view of (4.43), we get the expression for curvature tensor as

$$R(X, Y)Df = (\nabla_X Q)(Y) - (\nabla_Y Q)(X), \tag{4.44}$$

for all the vector fields X, Y on $M(f_1, f_2, f_3)$. We have from equation (4.44)

$$g(R(\xi, Y)Df, \xi) = g((\nabla_\xi Q)(Y) - (\nabla_Y Q)(\xi), \xi). \tag{4.45}$$

In consequence of (2.7) and Theorem 3.1, we can easily prove that

$$(\nabla_Y Q)(\xi) = 0. \tag{4.46}$$

From equation (2.7), we conclude that

$$(\nabla_X Q)(Y) = da(X)Y + db(X)\eta(Y)\xi + b[(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi].$$

Setting $X = \xi$ in last expression and utilizing the equations (2.4), (2.5), (3.13) and (3.25), we have

$$(\nabla_\xi Q)(Y) = da(\xi)(Y - \eta(Y)\xi). \tag{4.47}$$

By using the equations (4.46) and (4.47), equation (4.45) assumes the form

$$g(R(\xi, Y)Df, \xi) = 0, \quad \forall \ \ Y \in T(M). \tag{4.48}$$

Also from equations (2.9) and Theorem 3.1, we get

$$g(R(\xi, Y)Df, \xi) = (f_1 - f_3)\{g(Y, Df) - \eta(Df)\eta(Y)\}. \tag{4.49}$$

Equations (4.48) and (4.49) give

$$Df = (\xi f)\xi, \tag{4.50}$$

provided $f_1 \neq f_3$ on $M(f_1, f_2, f_3)$. In view of (4.43) and (4.50), we compute

$$S(X, Y) + \lambda g(X, Y) = g(Y(\xi f)\xi + (\xi f)\nabla_Y \xi, X). \tag{4.51}$$

Changing X with ξ in (4.51) and then using the equations (2.4), (2.5) and (2.10), we have

$$Y(\xi f) = \{\lambda + 2n(f_1 - f_3)\}\eta(Y). \tag{4.52}$$

Using (4.52) in (4.51), we find that

$$S(X, Y) + \lambda g(X, Y) = \{\lambda + 2n(f_1 - f_3)\}\eta(X)\eta(Y) + (\xi f)g(\nabla_Y \xi, X). \tag{4.53}$$

By the help of (4.53), equation (4.43) takes the form

$$\nabla_Y Df = \{\lambda + 2n(f_1 - f_3)\}\eta(Y)\xi + (\xi f)\nabla_Y \xi. \tag{4.54}$$

We have from equation (4.54)

$$\begin{aligned} R(X, Y)Df &= (\xi f)R(X, Y)\xi + [\lambda + 2n(f_1 - f_3)]\{d\eta(X, Y)\xi \\ &\quad +\eta(Y)\nabla_X \xi - \eta(X)\nabla_Y \xi\} + X(\xi f)\nabla_Y \xi - Y(\xi f)\nabla_X \xi, \end{aligned}$$

which gives

$$g(R(X, Y)(\xi f)\xi, \xi) = [\lambda + 2n(f_1 - f_3)]d\eta(X, Y), \tag{4.55}$$

where equations (2.4), (2.5), (2.8), (4.50) and (4.52) are used. From (4.55), we get

$$\lambda = -2n(f_1 - f_3) \tag{4.56}$$

because $d\eta(X,Y)$ is non-vanishing on contact metric manifold (in general). From (4.52) and (4.56), we calculate that

$$\xi f = c \quad (constant)$$

and hence

$$df = c\xi \Longrightarrow cd\eta = 0 \Longrightarrow c = 0.$$

Using this fact in above equation, we conclude that $f = constant$. In view of (4.53), (4.56) and above facts, we obtain

$$S(X,Y) = 2n(f_1 - f_3)g(X,Y).$$

If $f_1 \neq f_3$, then by Theorem 3.1 we can say that $M(f_1, f_2, f_3)$ is an Einstein manifold. Thus we state:

Theorem 4.2. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form whose Ricci tensor is cyclic parallel and $f_1 \neq f_3$. If the metric g of $M(f_1, f_2, f_3)$ is a gradient Ricci soliton, then the manifold is an Einstein manifold and the scalar curvature is constant.*

5. Examples

Example 5.1. *Let $N(p,q)$ be a generalized complex-space-form of dimension $2n$, then by the warped product $M = \Re \times_f N$ endowed with the almost contact metric structure (ϕ, ξ, η, g_f) is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with*

$$f_1 = \frac{p - (f')^2}{f^2}, \quad f_2 = \frac{q}{f^2}, \quad f_3 = \frac{p - (f')^2}{f^2} + \frac{f''}{f}, \tag{5.57}$$

where $f = f(t), t \in \Re$ (set of real number) and f' denotes the first derivative of f with respect to t and f'', second derivative of f with respect to t [1]. If we choose $f(t) = sin\, pt$ for non-zero constant p, where $t \neq \frac{2n\pi}{p}, \frac{\pi+2n\pi}{p}$ and $q = \frac{(2n-1)p(p-1)}{3}(\neq 0)$, then equation (5.57) takes the form

$$f_1 = \frac{p - p^2 cos^2 pt}{sin^2 pt}, \quad f_2 = \frac{(2n-1)(p^2 - p)}{3 sin^2 pt} \quad and \quad f_3 = \frac{p - p^2}{sin^2 pt}.$$

It is obvious from the above expression that $f_1 - f_3 = p^2 = constant$. Hence the Theorem 3.1 is verified.

Also, $(2n-1)f_1 + 3f_2 = (2n-1)p^2 = constant$ and $3f_2 + (2n-1)f_3 = 0$. Again from (2.11) and above relation, the scalar curvature $r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\} = (2n+1)p^2 = constant$. These relations verify the statement of the Lemma 3.1.

Above result with (2.6) give

$$S(X,Y) = 2np^2 g(X,Y), \tag{5.58}$$

for arbitrary vector fields X and Y on $M(f_1, f_2, f_3)$. Taking covariant derivative of (5.58) along the vector field Z, we have

$$(\nabla_Z S)(X,Y) = 0. \tag{5.59}$$

From equation (5.59), we can easily observe that $M(f_1, f_2, f_3)$ is Ricci symmetric if and only if $3f_2 + (2n-1)f_3 = 0$. Hence the statement of the Theorem 3.2.

Example 5.2. *Let us suppose that $M = \mathfrak{R} \times_f N$ equipped with an almost contact metric structure (ϕ, ξ, η, g_f) is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$, where f_1, f_2, f_3 are smooth functions on M defined in (5.57) and $N(p,q)$ be a generalized complex-space-form. If we consider $f(x) = e^{2t}$, $t \in \mathfrak{R}$ and $3q + (2n-1)p = \mu e^{4t}$ $(0 \neq \mu \in \mathfrak{R})$, then equation (5.57) converts in to the form*

$$f_1 = \frac{p - 4e^{4t}}{e^{4t}}, \quad f_2 = \frac{q}{e^{4t}}, \quad f_3 = \frac{p - 4e^{4t}}{e^{4t}} + 4. \tag{5.60}$$

It is obvious from equation (5.60), $f_1 - f_3 = -4(constant)$. Hence the statement of the Theorem 3.1.

In view of (2.11) and (5.60), we find that

$$(2n-1)f_1 + 3f_2 = \mu - 4(2n-1) = constant, \quad 3f_2 + (2n-1)f_3 = \mu = constant$$

and $r = 2n(\mu - 4(2n-1)) = constant$. From the above discussion, we can see that Lemma 3.1 is verified.

Also equations (2.11) and (5.60) give

$$S(X,Y) = (\mu - 8n)g(X,Y) - \mu\eta(X)\eta(Y), \quad \forall\, X, Y, Z \in T(M). \tag{5.61}$$

Differentiating (5.61) covariantly along the vector field Z, we obtain

$$(\nabla_Z S)(X,Y) = -\mu\{(\nabla_Z \eta)(X)\eta(Y) + (\nabla_Z \eta)(Y)\eta(X)\}. \tag{5.62}$$

From (5.62), we find that

$$\begin{aligned}
&(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) \\
&\quad = -\mu\{[(\nabla_Z \eta)(X) + (\nabla_X \eta)(Z)]\eta(Y) + [(\nabla_Z \eta)(Y) \\
&\quad + (\nabla_Y \eta)(Z)]\eta(X) + [(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X)]\eta(Z)\}.
\end{aligned} \tag{5.63}$$

From equation (5.63), it can be easily prove that the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel if and only if the characteristic vector field ξ is Killing. Thus the Theorem 3.5 is verified.

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